

Linear functions preserving Green's relations over fields

Artem Maksaev

Lomonosov Moscow State University,

Moscow Center for Fundamental and Applied Mathematics

tematematik@gmail.com

The talk is based on the joined work with A. Guterman, M. Johnson, and M. Kambites [5].

Green's relations are a number of equivalence relations and pre-orders which are defined upon any semigroup. Introduced by J. Green [1] in 1951, they encapsulate the ideal structure of the semigroup, and play a central role in almost every aspect of semigroup theory. The following are the definitions of Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{D}$.

Definition. Let \mathcal{M} be a monoid. For $a, b \in \mathcal{M}$, we say that:

- (i) $a \mathcal{R} b$ if $a\mathcal{M} = b\mathcal{M}$;
- (ii) $a \mathcal{L} b$ if $\mathcal{M}a = \mathcal{M}b$;
- (iii) $a \mathcal{J} b$ if $\mathcal{M}a\mathcal{M} = \mathcal{M}b\mathcal{M}$;
- (iv) $a \mathcal{H} b$ if $a \mathcal{R} b$ and $a \mathcal{L} b$;
- (v) $a \mathcal{D} b$ if there exists $c \in \mathcal{M}$: $a \mathcal{R} c$ and $c \mathcal{L} b$.

In particular, these are natural relations to define upon the set of $n \times n$ matrices over any semiring, when viewed as a semigroup under matrix multiplication.

The investigation of linear transformations preserving natural functions, invariants and relations on matrices has a long history, dating back to a result of Frobenius [2] describing maps which preserve the determinant. During the past century a lot of effort has been devoted to the development of this theory, particularly, for matrices over semirings.

In 2018, motivated by recent interest in the structure of the tropical semifield, A. Guterman, M. Johnson, and M. Kambites [3] characterized bijective linear maps which preserve (or strongly preserve) each of Green's relations on the space of $n \times n$ matrices over an anti-negative semifield. The results of [3] are rather unusual, since they provided a classification for all semifields except fields. Thus, the question arises, what the corresponding maps over fields are. The aim of this talk is to answer it.

Let \mathbb{F} be a field and $M_n(\mathbb{F})$ be the monoid of $n \times n$ matrices over \mathbb{F} . Based on a convenient description of Green's relations for $M_n(\mathbb{F})$, we present a complete classification of linear maps $T: M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ preserving each of these relations. However, some additional assumptions are needed, namely:

- for all relations $(\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{D})$, the classification was carried out under the assumption that T is bijective;
- for the relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$, it was done under the assumption that the field \mathbb{F} contains roots of all polynomials from $\mathbb{F}[x]$ of degree n (particularly, for algebraically closed field);
- for the relations \mathcal{J}, \mathcal{D} , no additional assumptions are needed.

Also, for the relation \mathcal{H} , it holds that non-zero \mathcal{H} -preservers coincide exactly with invertibility preservers, which classification is a well-known preserver problem. Over an arbitrary field, linear maps preserving invertibility have been fully described by de Seguins Pazzis [4].

Furthermore, we show some examples of non-bijective linear \mathcal{L} -, \mathcal{R} -, and \mathcal{H} -preservers that are not of the form mentioned in the classification. This shows the importance of the restrictions on the field or map stated above.

References

- [1] J. A. Green. On the structure of semigroups. *Annals of Math.* **54** (1951), 163-172.
- [2] G. Frobenius. Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, *Sitzungsber. Deutsch. Akad. Wiss.* (1897), 994-1015.
- [3] A. Guterman, M. Johnson, M. Kambites. Linear isomorphisms preserving Green's relations for matrices over anti-negative semifields, *Linear Algebra Appl.* **545** (2018), 1-14.
- [4] C. de Seguins Pazzis. The singular linear preservers of non-singular matrices, *Linear Algebra Appl.* **433** (2010), 483-490.
- [5] A. Guterman, M. Johnson, M. Kambites, A. Maksaev. Linear functions preserving Green's relations over fields, *Linear Algebra Appl.* **611** (2021), 310-333.