Closures of solvable permutation groups

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Let m be a positive integer and let Ω be a finite set. The *m-closure* $G^{(m)}$ of $G \leq Sym(\Omega)$ is the largest permutation group on Ω having the same orbits as G in its induced action on the Cartesian product Ω^m. Wielandt [5] showed that

$$G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(m)} = G^{(m+1)} = \cdots = G, \qquad (1)$$

for some $m < |\Omega|$. (Since the stabilizer in G of all but one point is always trivial, $G^{(n-1)} = G$ where $n = |\Omega|$.) In this sense, the m-closure can be considered as a natural approximation of G. It was shown by Praeger and Saxl [2] that for $m \geq 6$, the m-closure $G^{(m)}$ of a primitive permutation group G has the same socle as G. Furthermore, they classified explicitly primitive groups G and H with different socles having the same m-orbits for $m \leq 5$. Unfortunately, their results say very little about closures of solvable permutation groups. The main goal of this talk is to present the results of [1], where we study such closures.

The 1-closure of G is the direct product of symmetric groups $Sym(\Delta)$, where Δ runs over the orbits of G. Thus the 1-closure of a solvable group is solvable if and only if each of its orbits has cardinality at most 4. The case of 2-closure is more interesting. The 2-closure of every (solvable) 2-transitive group $G \leq Sym(\Omega)$ is $Sym(\Omega)$; other examples of solvable G and nonsolvable $G^{(2)}$ appear in [4]. But, as shown by Wielandt [5], each of the classes of finite p-groups and groups of odd order is closed with respect to taking the 2-closure. Currently, no characterization of solvable groups having solvable 2-closure is known.

Seress [3] observed that if G is a primitive solvable group, then $G^{(5)} = G$; so the 5-closure of a primitive solvable group is solvable. Our main result is the following stronger statement.

Theorem 1. *The* 3*-closure of a solvable permutation group is solvable.*

The corollary below is an immediate consequence of Theorem 1 and the chain of inclusions (1).

Corollary 2. *For every integer* $m \geq 3$*, the* m*-closure of a solvable permutation group is solvable.*

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References

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