Noncommutativity in an algebraic theory of clones

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We introduce the notion of clone algebra, intended to found a one-sorted, purely algebraic theory of clones. Clone algebras are defined by true identities and thus form a variety in the sense of universal algebra. The most natural clone algebras, the ones the axioms are intended to characterise, are algebras of functions, called functional clone algebras. The universe of a functional clone algebra, called omega-clone, is a set of infinitary operations containing the projections and closed under finitary compositions. We show that there exists a bijective correspondence between clones (of finitary operations) and a suitable subclass of functional clone algebras, called block algebras. Given a clone, the corresponding block algebra is obtained by extending the operations of the clone by countably many dummy arguments. One of the main results is the general representation theorem, where it is shown that every clone algebra is isomorphic to a functional clone algebra. In another result we prove that the variety of clone algebras is generated by the class of block algebras. This implies that every omega-clone is algebraically generated by a suitable family of clones by using direct products, subalgebras and homomorphic images. We present three applications. In the first one, we use clone algebras to answer a classical question about the lattices of equational theories. The second application is to the study of the category VAR of all varieties. We introduce the category CA of all clone algebras (of arbitrary similarity type) with pure homomorphisms as arrows. We show that the category VAR is categorically isomorphic to a full subcategory of CA. We use this result to provide a generalisation of a classical theorem on independent varieties. In the third application we show how skew Boolean algebras are related to clone algebras.