SEIBERG–WITTEN EQUATIONS AND PSEUDOHOLOMORPHIC CURVES

Armen SERGEEV

Steklov Mathematical Institute, Moscow

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The Seiberg–Witten equations, discovered at the end of XXth century, became one of main inventions in the topology of smooth 4-manifolds. Along with Yang–Mills equations they are the limiting case of a general supersymmetric Yang–Mills theory. However, opposite to conformally invariant Yang–Mills equations, the Seiberg–Witten equations are not invariant under the scale change.

So, in order to draw "useful information" from these equations one has to introduce a scale parameter λ into them and consider the limit $\lambda \to \infty$. This is precisely the adiabatic limit which we study here.

I. Seiberg–Witten equations

In order to define the Seiberg–Witten equations I have to use some basic notions from spinor geometry.

Let (X, g) be a compact oriented Riemannian 4-manifold provided with Levi-Civita connection. Then we can define the Clifford multiplication ρ, i.e. a representation of forms from $\Omega^*(X)$ by linear endomorphisms acting on smooth sections of the spinor bundle $W \to X$. It is a complex Hermitian vector bundle of rank 4 decomposed into the direct sum

$$W = W^+ \oplus W^-$$

of semispinor bundles of rank 2.

The spinor bundle W may be provided with the spinor connection ∇ being an extension of the Levi-Civita connection to a connection on W. The Dirac operator on smooth sections of W is given by the composition $\rho \circ \nabla$ of Clifford multiplication with spinor connection.

In case when the manifold (X, g) is symplectic, i.e. provided with a symplectic form ω compatible with g, it has also an almost complex structure J compatible both with ω and g.

In this case we have a canonical construction of the spinor bundle W identified with

$$W_{\mathrm{can}} = \Lambda^{0,*}(T^*X) = \bigoplus_{q=0}^{2} \Lambda^{0,q}(T^*X).$$

Accordingly,

$$W^{+}_{\mathrm{can}} = \Lambda^{0,0}(T^*X) \oplus \Lambda^{0,2}(T^*X)\,, \quad W^{-}_{\mathrm{can}} = \Lambda^{0,1}(T^*X).$$

There is also an explicit formula for the Clifford multiplication and a canonical construction of the spinor connection on W_{can}.

Moreover, for any Hermitian line bundle $E \to X$ with a Hermitian connection B on it we can construct an associated spinor bundle $W_E := W_{\mathrm{can}} \otimes E$ and a spinor connection ∇_A on W_E where $A = A_E$ is the tensor product of the canonical spinor connection on W_{can} and given Hermitian connection B on E.

The Dirac operator

$$D_A = \rho \circ \nabla_A : \Gamma(X, W^+) \longrightarrow \Gamma(X, W^-)$$

coincides in this case with $\bar{\partial}_B + \bar{\partial}_B^*$ where $\bar{\partial}_B^*$ is the L^2-adjoint of the operator $\bar{\partial}_B$.

Introduce the **Seiberg–Witten action** functional

$$S(A,\Phi) = \frac{1}{2}\int_X \left\{ |F_A|^2 + |\nabla_A \Phi|^2 + \left(s(g) + |\Phi|^2\right)\frac{|\Phi|^2}{4} \right\} \mathrm{vol}$$

where $s(g)$ is the scalar curvature of (X, g), F_A is the curvature of ∇_A, $\Phi \in \Gamma(X, W)$ is a smooth section of spinor bundle W and vol is the volume element on (X, g).

The local minima of this functional satisfy the **Seiberg–Witten equations**

$$\begin{cases} D_A \Phi = 0, \\ F_A^+ = \Phi \otimes \Phi^* - \dfrac{1}{2}|\Phi|^2 \cdot \mathrm{Id} \end{cases}$$

where F_A^+ is the selfdual component of the curvature F_A.

Solutions of these equations are given by pairs (A, Φ) where $A = A_E$ was described above and Φ is a section of semispinor bundle W^+ represented by the pair of forms (φ_0, φ_2) with $\varphi_0 \in \Omega^0(X, E)$, $\varphi_2 \in \Omega^{0,2}(X, E)$.

In fact, Seiberg–Witten theory depends essentially only on the complex line bundle $E \to X$ provided with Hermitian connection B and in this sense this theory is Abelian.

II. Ginzburg–Landau vortices

The Seiberg–Witten equations may be considered as a 4-dimensional generalization of static Ginzburg–Landau equations in the plane $\mathbb{R}^2$. In this case the Seiberg–Witten action functional is reduced to the potential energy functional

$$U(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2} = \left\{|F_A|^2 + |d_A\Phi|^2 + \frac{1}{4}(1-|\Phi|^2)^2\right\} dx_1 dx_2$$

where A is a U(1)-connection on $\mathbb{R}^2$ represented in coordinates (x_1, x_2) by the 1-form $A = A_1 dx_1 + A_2 dx_2$ with smooth pure imaginary coefficients. The curvature of A is given by tyhe formula

$$F_A = dA = \sum_{i,j=1}^{2} F_{ij} dx_i \wedge dx_j$$

and $\Phi = \Phi_1 + i\Phi_2$ is a smooth complex-valued function on $\mathbb{R}^2_{(x_1,x_2)}$.

We require that $|\Phi| \to 1$ for $|x| \to \infty$ which implies that our problem has an integer topological invariant, called the vortex number d, given by the rotation number of the map Φ sending the circles of sufficiently large radius to topological circles.

We look for the pairs (A, Φ), minimizing the potential energy $U(A, \Phi)$ in a given topological class, fixed by the value of d. These pairs satisfy (for $d > 0$) the system of vortex equations analogous to Seiberg–Witten equations in dimension 4.

In the complex coordinate $z = x_1 + ix_2$ they have the form

$$\begin{cases} \bar{\partial}_A \Phi = 0, \\ iF_{12} = \frac{1}{2}(1 - |\Phi|^2), \end{cases}$$

where $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ for $A = A^{1,0} + A^{0,1}$.

These equations, as well as potential energy $U(A, \Phi)$, are invariant under gauge transforms given by:

$$A \mapsto A + id\chi, \quad \Phi \mapsto e^{-i\chi}\Phi$$

with arbitrary smooth real-valued function χ on $\mathbb{R}^2_{(x_1,x_2)}$.

We call solutions of vortex equations d-vortices and are interested in their moduli space, i.e. the space $\mathcal{M}_d$ of d-vortices (A, Φ) modulo gauge transforms. In terms of complex coordinate $z = x_1 + ix_2$ this moduli space is described by the following theorem of Taubes.

Theorem of Taubes

For any unordered collection $z_1, \ldots, z_k$ of k points on the complex plane $\mathbb{C}$, taken with multiplicities $d_1, \ldots, d_k$ such that $d_1 + \ldots + d_k = d$, there exists a unique (up to gauge transforms) d-vortex (A, Φ) such that the map Φ vanishes precisely at the points $z_1, \ldots, z_k$ with given multiplicities $d_1, \ldots, d_k$.

Moreover, Taubes has proved that any critical point (A, Φ) of the functional $U(A, \Phi) < \infty$ with vortex number $d > 0$ is gauge equivalent to some d-vortex. In other words, all solutions of the Euler–Lagrange equations for the functional $U(A, \Phi)$ with finite energy have minimal energy in their topological class.

This theorem implies that the moduli space $\mathcal{M}_d$ of d-vortices may be identified with complex vector space $\mathbb{C}^d$ by assigning to the collection $z_1, \ldots, z_d$ the monic polynomial having its zeros precisely at given points $z_1, \ldots, z_d$ with given multiplicities.

III. Adiabatic limit in Ginzburg–Landau equations

The dynamics of vortices is described by the Ginzburg–Landau action functional on $\mathbb{R}^3_{(t,x_1,x_2)}$ given by the formula

$$S(\mathcal{A},\Phi) = \int_0^{T_0} \left(T(\mathcal{A},\Phi) - U(\mathcal{A},\Phi)\right) dt$$

where the 1-form A is replaced by the 1-form

$$\mathcal{A} = A_0 dt + A_1 dx_1 + A_2 dx_2$$

with smooth pure imaginary coefficients $A_\mu = A_\mu(t, x_1, x_2)$, $\mu = 0, 1, 2$, and $\Phi = \Phi(t, x_1, x_2)$ being a smooth complex-valued function on $\mathbb{R}^3_{(t,x_1,x_2)}$. Denote by $A^0 = A_0 dt$ the time component of $\mathcal{A}$ and by $A = A_1 dx_1 + A_2 dx_2$ its space component.

Then the potential energy of the system will be given by the same formula, as before, i.e. $U(\mathcal{A}, \Phi) = U(A, \Phi)$, while the kinetic energy has the form

$$T(\mathcal{A}, \Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |F_{01}|^2 + |F_{02}|^2 + |d_{A^0}\Phi|^2 \right\} dx_1 dx_2$$

where $F_{0j} = \partial_0 A_j - \partial_j A_0$ for $j = 1, 2$ and $d_{A^0}\Phi = d\Phi + A_0\, dt$.

The Ginzburg–Landau action $S(\mathcal{A}, \Phi)$ is invariant under dynamical gauge transforms, given by the same formula as before with gauge function $\chi = \chi(t, x_1, x_2)$ depending also on time.

We are interested in the critical points of the Ginzburg–Landau action functional $S(\mathcal{A}, \Phi)$, i.e. solutions of the corresponding Euler–Lagrange equations, called briefly dynamical solutions, modulo dynamical gauge transforms.

It is convenient to choose the gauge function $\chi(t, x_1, x_2)$ so that the time component of the potential will vanish, i.e. $A_0 = 0$ (temporal gauge). After imposing this condition on gauge function χ we are still left with the gauge freedom with respect to static gauge transforms given by gauge functions $\chi(x_1, x_2)$ not depending on time.

In this gauge we can consider a dynamical solution as a trajectory of the form $\gamma : t \longmapsto [A(t), \Phi(t)]$ where $[A, \Phi]$ denotes the gauge class of (A, Φ) with respect to static gauge transforms.

This trajectory lies in the configuration space

$$\mathcal{N}_d = \frac{\{(A, \Phi) \text{ with } U(A, \Phi) < \infty \text{ and vortex number } d\}}{\{\text{static gauge transforms}\}}.$$

The configuration space $\mathcal{N}_d$ may be thought of as a canyon with the bottom occupied by the moduli space $\mathcal{M}_d$ of d-vortex solutions having minimal potential energy.

Then a dynamical solution may be considered as the trajectory $\gamma(t)$ of a small ball rolling along the walls of canyon. The lower is kinetic energy of the ball, the closer is its trajectory to the bottom. Our ball may even hit the bottom but cannot stop there since, having a non-zero kinetic energy, it should assent the canyon wall again.

Consider a family of dynamical solutions $\gamma_\epsilon : t \longmapsto [A_\epsilon(t), \Phi_\epsilon(t)]$ of Ginzburg–Landau equations depending on a parameter $\epsilon > 0$ with vortex number d. Suppose that the kinetic energy of these trajectories

$$T(\gamma_\epsilon) = \int_0^{T_0} T(\gamma_\epsilon(t))dt$$

tends to zero for $\epsilon \to 0$ proportional to ϵ. Then in the limit $\epsilon \to 0$ the trajectory γ_ϵ converts into a static solution, i.e. a point of $\mathcal{M}_d$.

However, if we introduce a "slow time" parameter $\tau = \epsilon t$ on γ_ϵ and consider the limit of the "rescaled" trajectories $\gamma_\epsilon(\tau)$ for $\epsilon \to 0$ then in this limit we shall obtain a trajectory γ_0, lying in $\mathcal{M}_d$, rather than a point.

This procedure is called the adiabatic limit and the original Ginzburg–Landau equations for $S(\mathcal{A}, \Phi)$ in this limit reduce to the adiabatic equations whose solutions are called adiabatic trajectories.

The adiabatic trajectories admit the following intrinsic description in terms of the moduli space $\mathcal{M}_d$. More precisely, we have the following

Theorem

Kinetic energy functional generates a Riemannian metric on the space $\mathcal{M}_d$, called kinetic or T-metric. Adiabatic trajectories γ_0 are geodesics of this metric.

The idea of approximate description of "slow" dynamical solutions in terms of the moduli space of static solutions was proposed on an heuristic level by Manton who postulated the following adiabatic principle: for any geodesic trajectory γ_0 on the moduli space of d-vortices $\mathcal{M}_d$ there exists a sequence $\{\gamma_\epsilon\}$ of dynamical solutions, converging to γ_0 in the adiabatic limit.

A rigorous mathematical formulation and the proof of this principle are given by Roman Palvelev.

Adiabatic principle reduces the description of vortex dynamics to the description of geodesics on the moduli space of d-vortices $\mathcal{M}_d$ in the kinetic metric, i.e. to the solution of Euler geodesic equation on the space $\mathcal{M}_d$ provided with T-metric.

IV. Adiabatic limit in Seiberg–Witten equations on symplectic 4-manifolds

We return now to the case of 4-dimensional symplectic Riemannian manifold (X, g) provided with symplectic 2-form ω and compatible almost complex structure J. As in the 2-dimensional case the Seiberg–Witten equations, as well as the Seiberg–Witten action, are invariant under gauge transforms given by the similar formula with gauge function $g = e^{i\chi} \in C^\infty(X, \mathrm{U}(1))$.

I recall that the Seiberg–Witten equations on X have the form

$$\begin{cases} D_A \Phi = 0, \\ F_A^+ = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \cdot \mathrm{Id}. \end{cases}$$

where F_A^+ is the selfdual component of the curvature F_A.

We denote as before by $E \to X$ a Hermitian line bundle with a Hermitian connection B and by $W_E := W_{\text{can}} \otimes E$ is the spinor bundle provided with a spinor connection ∇_A on where $A = A_E$ is the tensor product of the canonical spinor connection on W_{can} and connection B on E.

In order to guarantee the solvability of these equations we should impose on the 1st Chern class $c_1(E)$ certain topological bounds and consider a perturbation of these equations obtained by plugging an appropriate selfdual 2-form η into the second equation.

The complexified bundle $\Lambda^2_+ \otimes \mathbb{C}$ of selfdual 2-forms on X in the considered case is decomposed into the direct sum of subbundles

$$\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{2,0} \oplus \mathbb{C}[\omega] \oplus \Lambda^{0,2}.$$

Accordingly, the second Seiberg–Witten equation for the curvature decomposes into the sum of three equations — the one for the component, parallel to ω, the $(0,2)$-component and $(2,0)$-component which is conjugate to $(0,2)$-component and by this reason is omitted below.

The Seiberg–Witten equations take the form

$$\begin{cases} \bar{\partial}_B \varphi_0 + \bar{\partial}_B^* \varphi_2 = 0, \\ F_B^{0,2} + \eta^{0,2} = \dfrac{\bar{\varphi}_0 \varphi_2}{2}, \\ F_{A_{\mathrm{can}}}^{\omega} + F_B^{\omega} = \dfrac{i}{4}(|\varphi_0|^2 - |\varphi_2|^2) - \eta^{\omega}. \end{cases}$$

We introduce into the Seiberg–Witten equations the scale parameter $\lambda > 0$ and take the perturbation η in the form $\eta = -F_{A_{\mathrm{can}}}^{+} + \pi i \lambda \omega$.

In terms of renormalized sections $\alpha := \frac{\varphi_0}{\sqrt{\lambda}}$ and $\beta := \frac{\varphi_2}{\sqrt{\lambda}}$ the perturbed Seiberg–Witten equations will rewrite as

$$\begin{cases} \bar{\partial}_B \alpha + \bar{\partial}_B^* \beta = 0, \\ \frac{2}{\lambda} F_B^{0,2} = \bar{\alpha}\beta, \\ \frac{4i}{\lambda} F_B^{\omega} = 4\pi + |\beta|^2 - |\alpha|^2. \end{cases}$$

According to Taubes, solutions $(\alpha_\lambda, \beta_\lambda)$ of these equations have the following behavior for $\lambda \to \infty$:

(1) $|\alpha_\lambda| \to 1$ everywhere outside the set of zeros $\alpha_\lambda^{-1}(0)$;

(2) $|\beta_\lambda| \to 0$ everywhere together with 1st order derivatives.

Denote by $C_\lambda := \alpha_\lambda^{-1}(0)$ the zero set of α_λ. The curves C_λ converge in the sense of currents to some pseudoholomorphic divisor, i.e. a chain $\sum d_k C_k$, consisting of connected pseudoholomorphic curves C_k taken with miltiplicities d_k.

Simultaneously, the original Seiberg–Witten equations reduce to a family of vortex equations in the complex planes normal to the curves C_k. The chain $\sum d_k C_k$ may be considered as a complex analogue of adiabatic geodesics in (2+1)-dimensional case.

Conversely, in order to reconstruct the solution of Seiberg–Witten equations from this family of vortex solutions in normal planes it should satisfy a nonlinear $\bar{\partial}$-equation which may be considered as a complex analogue of the Euler equation for adiabatic geodesics with "complex time".

Thus, for the Seiberg–Witten equations on symplectic 4-manifolds we have the following correspondence, established by the adiabatic limit:

$$\left\{\begin{array}{l}\textbf{solutions of}\\ \textbf{Seiberg–Witten}\\ \textbf{equations}\end{array}\right\} \longmapsto \left\{\begin{array}{l}\textbf{families of vortex solutions}\\ \textbf{in normal planes of pseudo-}\\ \textbf{holomorphic divisors}\end{array}\right\}$$