Normal Subgroup Based Power Graph of a Finite Group

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Power Graph and Power Digraph

Suppose G is a semigroup. The directed power graph $\overrightarrow{P}(G)$ has G as its vertex set and we have an arc from u to v if and only if v is a power of u. The undirected power graph $\mathcal{P}(G)$ of a semigroup G can be defined in a similar way. These graphs first studied by Kelarev and Quinn.

A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, Contributions to General Algebra 12 (Vienna, 1999), 229–235, Heyn, Klagenfurt, 2000.

A. V. Kelarev, S. J. Quinn and R. Smolikova, Power graphs and semigroups of matrices, Bulletin of the Australian Mathematical Society 63 (2) (2001) 341–344.

Normal Subgroup Based Power Graph

Suppose G is a group containing a normal subgroup H. The normal subgroup based power graph of G with respect to H is an undirected graph in which the vertex set is $(G \setminus H) \cup \{e_G\}$ and two distinct vertices x and y are adjacent if and only if $xH = y^m H$ or $yH = x^n H$, for some integers m and n.

A. K. Bhuniya and S. Bera, Normal subgroup based power graphs of a finite group, Communications in Algebra 45 (8) (2017) 3251–3259.

Cyclomatic Number of Graphs

The cyclomatic number of a connected graph Γ is defined as $c(\Gamma) = m - n + 1$, where m and n are the number of edges and the number of vertices, respectively. It is well-known that this is the number of independent cycles in Γ. In particular, the graphs with cyclomatic number $c = 0, 1, 2, 3, 4$ are called trees, unicyclic, bicyclic, tricyclic and tetracyclic graphs, respectively.

B. Bollobás, Graph Theory, An Introductory Course, Springer Verlag, New York, 1990.

Split, Bisplit and $k-$Bisplit Graphs

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. The graph Γ is called a bisplit graph if $V(\Gamma)$ can be partitioned into three independent subsets X, Y and Z such that $Y \cup Z$ induces a complete bipartite subgraph in Γ. The graph Γ is said to be $k-$bisplit if there exists an independent subset X of $V(\Gamma)$ such that the induced subgraph by $V \setminus X$ has at most k connected components in which all of components are complete bipartite graphs.

S. Foldes and P. L. Hammer, Split graphs having Dilworth number two, Canadian Journal of Mathematics 29 (3) (1977) 666–672.

Union of Graphs

Suppose Γ and Δ are two graphs with disjoint vertex sets. The graph union $\Gamma \cup \Delta$ is a graph with vertex set $V(\Gamma) \cup V(\Delta)$ and edge set $E(\Gamma) \cup E(\Delta)$. The graph union of r graphs with mutually disjoint vertex sets can be defined by induction. If these r graphs are isomorphic to a given graph Δ it is usual to use the notation of $r\Delta$ for the resulting graph.

Cartesian, Tensor and Strong Product of Graphs

Suppose Γ and Δ are two graphs. The *Cartesian product* $\Gamma \Box \Delta$ is a graph with vertex set $V(\Gamma \Box \Delta) = V(\Gamma) \times V(\Delta)$ in which two vertices (a, b) and (x, y) are adjacent if and only if ($a = x$ and $by \in E(\Delta)$) or ($ax \in E(\Gamma)$ and $b = y$). The *tensor product* $\Gamma \times \Delta$ has again $V(\Gamma) \times V(\Delta)$ as its vertex set. Two vertices (a, b) and (x, y) in $\Gamma \times \Delta$ are adjacent in if and only if $ax \in E(\Gamma)$ and $by \in E(\Delta)$. The *strong product* $\Gamma \boxtimes \Delta$ is a graph with vertex set $V(\Gamma \boxtimes \Delta) = V(\Gamma) \times V(\Delta)$ and edge set $E(\Gamma \Box \Delta) \cup E(\Gamma \times \Delta)$.

Some Graph Isomorphisms

It is easy to see that $\Gamma_1 \boxtimes \Gamma_2 \cong \Gamma_2 \boxtimes \Gamma_1$, $\Gamma_1 \boxtimes (\Gamma_2 \boxtimes \Gamma_3) \cong (\Gamma_1 \boxtimes \Gamma_2) \boxtimes \Gamma_3$ and $\Gamma_1 \boxtimes (\Gamma_2 \cup \Gamma_3) \cong \Gamma_1 \boxtimes \Gamma_2 \cup \Gamma_1 \boxtimes \Gamma_3$. Furthermore, $K_n \boxtimes K_m \cong K_{nm}$, where n and m are positive integers.

Suppose Γ and Δ are graphs with exactly m and n connected components, respectively. Then $\Gamma \boxtimes \Delta$ is a graph with exactly mn connected components. Note that if $\Gamma = \cup_{i=1}^{n} \Gamma_i$ and $\Delta = \cup_{j=1}^{m} \Delta_j$, then $\Gamma \boxtimes \Delta = \cup_{j=1}^{m} \cup_{i=1}^{n} (\Gamma_i \boxtimes \Delta_j)$ and each subgraph $\Gamma_i \boxtimes \Delta_j$ is a connected component of the whole graph.

H-Join of Graphs

Consider a family $\Gamma_1, \ldots, \Gamma_k$ of graphs and a given graph H with vertex set $\{1, 2, \ldots, k\}$. The $H-$join of $\Gamma_1, \ldots, \Gamma_k$, $H[\Gamma_1, \ldots, \Gamma_k]$, is a graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2) \cup \ldots \cup V(\Gamma_k)$ and edge set $X \cup Y$, where $X = E(\Gamma_1) \cup E(\Gamma_2) \cup \ldots \cup E(\Gamma_k)$ and Y is the set of all edges uv such that $u \in V(G_r)$, $v \in V(G_s)$, $rs \in E(H)$ and $1 \leq r \neq s \leq k$.

Power Graph of Groups

Let G be a group. Then the following hold:

1. The power graph $\mathcal{P}(G)$ is connected if and only if every element of G is of finite degree, i.e., G is a periodic group.
2. Let G be a finite group. Then $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and for some positive integer m.
3. Let G be a finite group of order n. Then $\mid E(\mathcal{P}(G)) \mid = \frac{1}{2}\sum_{d|n}[2d - \phi(d) - 1]\phi(d)$, where $\phi(d)$ denotes the Euler totient function.

I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, Semigroup Forum, 78 (2009) 410–426.

Power Graph of Infinite Groups

Let G be a group. Then for torsion-free nilpotent groups of class at most 2, and for groups in which every non-identity element lies in a unique maximal cyclic subgroup, the power graph determines the directed power graph up to isomorphism.

Peter J. Cameron, H. Guerra, S. Jurina, The power graph of a torsion-free group, Journal of Algebraic Combinatorics 49 (2019) 83–98.

Main Properties of $\mathcal{P}_H(G)$

Let G be a finite group containing a normal subgroup H. Then the following hold:

1. Suppose aH and bH are two distinct cosets of H, where $aH \neq H$ and $bH \neq H$. If an element of aH is adjacent with an element of bH, then each element of aH is adjacent with every element of bH.
2. Suppose a and b are two distinct vertices of the graph $\mathcal{P}_H(G)$. Then a is adjacent with b in $\mathcal{P}_H(G)$ if and only if either $aH = bH$ or aH is adjacent with bH in the power graph $\mathcal{P}(\frac{G}{H})$.
3. $\mathcal{P}^\star_H(G) \cong \mathcal{P}^\star(\frac{G}{H}) \boxtimes K_{|H|}$.
4. The graph $\mathcal{P}_H(G)$ is complete if and only if $\frac{G}{H}$ is a cyclic $p-$group, for a prime number p. Furthermore, this graph has at least one $3-$cycle.
5. If $1 < |H| < |G|$, then the graph $\mathcal{P}_H(G)$ is neither bipartite nor tree.

A. K. Bhuniya and S. Bera, Normal subgroup based power graphs of a finite group, Communications in Algebra, 45 (8) (2017) 3251–3259.

Groups of Order pq and Elementary Abelian Groups

Suppose ϕ denotes the Euler totient function and $T_{p,q}$ is the unique non-abelian group of order pq, where p and q are distinct primes and $p|q-1$. Then the following hold:

1. Suppose p and q are two distinct primes, where $p < q$. Then $\mathcal{P}(Z_{pq}) \cong (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)+1}$ and if $p|q-1$ then $\mathcal{P}(T_{p,q}) \cong K_1 + (qK_{p-1} \cup K_{q-1})$.
2. Let G be an elementary abelian group of order p^n, where p is prime and n is a positive integer. Then $\mathcal{P}(G) \cong K_1 + lK_{p-1}$, where $l = \frac{p^n-1}{p-1}$.

T. Tamizh Chelvam and M. Sattanathan, Power graphs of finite abelian groups, *Algebra and Discrete Math.* **16** (1) (2013) 33–41.

Characterization of Split Graphs

A graph is split if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs, C_4, C_5 or $2K_2$.

S. Foldes and P. L. Hammer, Split graphs, in: Proceedings of the 8th South-Eastern Conference on Combinatorics, Graph Theory and Computing, 1977, pp. 311–315.

Split Power Graphs

For any group G, the power graph $\mathcal{P}(G)$ is split if and only if it has no induced subgraph isomorphic to the forbidden graph $2K_2$ if and only if G is isomorphic to one of the following groups:

1. An elementary abelian $2-$group, a cyclic $p-$group or a cyclic group of order $2p$;
2. A non-abelian $2-$group containing a maximal subgroup $M = \langle x \rangle$ such that (i) every element $y \in G \setminus M$ is an involution and $\langle y \rangle$ is not normal in G, (ii) M is normal in G and $\frac{G}{M}$ is elementary abelian, and (iii) for all $y \in G \setminus M$, $yxy = x^{-1}$.
3. A non-abelian group of type $Z_2^m \ltimes Z_{p^n}$ such that (i) every element $y \in G \setminus M$ is an involution and $\langle y \rangle$ is not normal in G, and (ii) for all $y \in G \setminus M$, $yxy = x^{-1}$.

X. Ma and M. Feng, On the chromatic number of the power graph of a finite group, Indagationes Mathematicae, 26 (4) (2015) 626–633.

Bipartite Graphs

Let G be a finite group. The power graph $\mathcal{P}(G)$ is bipartite if and only if G is an elementary abelian group of even order.

M. Mirzargar, A. R. Ashrafi, M. J. Nadjafi-Arani, On the Power Graph of a Finite Group, Filomat 26 (6) (2012) 1196–1203.

Theorem 1

Let G be a finite group, $1 \neq H \trianglelefteq G$ and $|G : H| = n$. The normal subgroup based power graph $\mathcal{P}_H(G)$ is a split graph if and only if $\mathcal{P}_H(G)$ is complete. The graph $\mathcal{P}_H(G)$ is bisplit if and only if $|H| = 2$ and $\frac{G}{H} \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. The graph $\mathcal{P}_H(G)$ is $(n-1)-$bisplit if and only if it is a bisplit graph.

Theorem 2

Let G be a finite group and $1 \neq H \lhd G$. Then the following hold:

1. the graph $\mathcal{P}_H(G)$ is unicyclic if and only if $|H| = 2$ and $G \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$;
2. the graph $\mathcal{P}_H(G)$ cannot be bicyclic or tetracyclic;
3. the graph $\mathcal{P}_H(G)$ is tricyclic if and only if $|H| = 2$ and $\frac{G}{H} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|H| = 3$ and $\frac{G}{H} \cong \mathbb{Z}_2$.

Theorem 3

Let G be a finite group and $1 \neq H \lhd G$. Then the graph $\mathcal{P}_H^\star(G)$ is bipartite if and only if $|H| = 2$ and $\frac{G}{H} \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Theorem 4

Let G be a finite group and H be a proper non-trivial normal subgroup of G. Then $diam(\mathcal{P}_H(G)) \leq 2$ with equality if and only if $\frac{G}{H}$ is not a cyclic $p-$group. Moreover, $diam(\mathcal{P}^{\star}_H(G)) = diam(\mathcal{P}^{\star}(\frac{G}{H}))$.

Theorem 5

Let G be a finite group and H be a proper non-trivial normal subgroup of G. The graph $\mathcal{P}^{\star}_{H}(G)$ is regular if and only if the proper power graph $\mathcal{P}^{\star}(\frac{G}{H})$ is regular.

Theorem 6

Let G be a finite group with a proper non-trivial normal subgroup H. We also assume that p and q are distinct primes. Then the following hold:

1. $\frac{G}{H} \cong T_{p,q}$ if and only if $\mathcal{P}_H(G) \cong K_1 + (qK_{(p-1)|H|} \cup K_{(q-1)|H|})$,
2. If $\frac{G}{H}$ is an elementary abelian group then $\mathcal{P}_H^{\star}(G)$ is a union of complete graphs.

Theorem 7

Let G be a finite group containing a proper non-trivial normal subgroup H. Then the following hold:

1. The graph $\mathcal{P}_H(G)$ is a line graph if and only if the graph is complete.
2. The graph $\mathcal{P}_H^{\star}(G)$ is a line graph if and only if each connected component of this graph is complete.

Theorem 8

Suppose G is a finite non-cyclic group. The power graph $\mathcal{P}(G)$ is a union of cliques with identity as the unique common vertex if and only if G is a union of cyclic subgroups with prime power orders and trivial intersection.

Theorem 9

Suppose G is a finite non-cyclic group and H is a proper normal subgroup of G. The normal subgroup based power graph $\mathcal{P}_H(G)$ is a union of cliques with identity as the unique common vertex if and only if $\frac{G}{H}$ is a union of cyclic subgroups with prime power orders and trivial intersection.

Three Important Questions about Power Graphs

(1) Is there any classification of pairs (G, H) for which $\mathcal{P}_H(G)$ is Hamiltonian?

(2) Which groups have the minimum number of edges in their power graphs?

(3) Suppose Γ is a given graph. How many groups there are with the power graph isomorphic to Γ?

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