A few new triplanes

(joint work with Dean Crnković)

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Abstract

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence I is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. We consider triplanes, i.e. symmetric block designs with $\lambda = 3$. Triplanes of order 12, i.e. symmetric (71,15,3) designs, have the greatest number of points among all known triplanes and it is not known if a triplane $(v, k, 3)$ exists for $v > 71$.

In this talk, in addition to reviewing previously known results, we give the first example of a triplane of order 12 that doesn't admit an automorphism of order 3.

Known triplanes for $v \leq 100$ and $k \leq \frac{v}{2}$[1]

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 3752
71	15	3	≥ 72
81	16	3	?

P.B.Gibbons, Computing Techniques for the Construction and Analysis of Block Designs, PhD Thesis, University of Toronto, 1976.

R.H.F.Denniston, Enumeration of symmetric designs (25,9,3), Ann. Discrete Math. 15 (1982), 111-127.

E. Spence, A complete classification of symmetric (31,10,3) designs, Des. Codes Crypt. 2 (1992), 127-136.

R.A.Mathon, E. Spence, On 2-(45,12,3) designs, J. Combin. Des. 4 (1996), 155-175.

W.H.Haemers, Eigenvalue Techniques in Design and Graph Theory, Mathematisch Centrum, Amsterdam, 1980.

S. Rukavina, Some new triplanes of order twelve, Glas. Mat. Ser. III 36(56) (2001), 105-125

[1] R. Mathon, A.Rosa, 2-(v, k, λ) Designs of Small Order, in: Handbook of Combinatorial Designs, 2^{nd} ed. (C. J. Colbourn and J. H. Dinitz, Eds.), Chapman and Hall/CRC, Boca Raton, 2007, 25–57.

Triplanes of order nine - CRC, 2007

- R. A. Mathon and E. Spence, On 2-$(45, 12, 3)$ designs, J. Combin. Des. 4 (1996), 155–175
  - R. M. MacFarland, A family of diference sets in non-cyclic groups, J. Comb. Th., A 15 (1973) 1–10.
  - T. Kölmel, Einbettbarkheit symmetrischer $(45, 12, 3)$ Blockplaene mit fixpunktfrei operierenden Automorphismen, Heidelberg 1991.
  - V. Ćepulić, On symmetric block designs $(45, 12, 3)$ with automorphism of order 5, Ars Combin., 37 (1994) 37-48.

- 1136 designs with trivial automorphism, under the assumption that the incidence matrix of a 2-$(45, 12, 3)$ design has a certain block structure
- all 2-$(45, 12, 3)$ designs having an automorphism of order 5 or 11
- the total number of non-isomorphic 2-$(45, 12, 3)$ designs found is 3752

Triplanes of order nine

- D. Crnković, D. Dumičić Danilović, SR, Enumeration of symmetric $(45, 12, 3)$ designs with nontrivial automorphisms, J. Algebra Comb. Discrete Struct. Appl. 3 (2016), 145–154.
  - U. Dempwolff, Primitive rank 3 groups on symmetric designs, Des. Codes Cryptogr. 22(2) (2001), 191–207.
  - D. Crnković, SR, M. Schmidt, A Classification of all Symmetric Block Designs of Order Nine with an Automorphism of Order Six, J. Combin. Des. 14, No.4 (2006), 301–312.
  - K. Coolsaet, J. Degraer, E. Spence, The strongly regular $(45, 12, 3, 3)$ graphs, Electron. J. Combin. 13 (2006), no. 1, Research Paper 32, 9 pp.

- There are exactly 4285 symmetric $(45, 12, 3)$ designs that admit nontrivial automorphisms. Among them there are 1161 self-dual designs and 1562 pairs of mutually dual designs.

Triplanes of order twelve - CRC, 2007

- W. H. Haemers, Eigenvalue Techniques in Design and Graph Theory, Mathematisch Centrum, Amsterdam, 1980.
- SR, Some new triplanes of order twelve, Glas. Mat. Ser. III 36(56) (2001), 105–125.
    - M. Garapić, Construction of some new triplanes. Ph. D. Thesis, University of Zagreb, 1993.

- The first triplane of order twelve has been constructed from an embeddable 2-$(56, 12, 3)$ design. Haemers constructed four mutually non-isomorphic symmetric $(71, 15, 3)$ designs which are not self-dual. Three of the constructed triplanes have the full automorphism group of order 336 and the order of the full automorphism group of the forth triplane is 48.
- 72 designs with an automorphism of order six acting with one fixed point

Triplanes of order twelve

- D. Crnković, SR, L. Simčić, On triplanes of order twelve admitting an automorphism of order six and their binary and ternary codes, Util. Math. 103 (2017), 23–40.
  - SR, 2-(56,12,3) designs and their class graphs, Glas. Mat. Ser. III 38(58) (2003), 201-210.

- all triplanes of order twelve admitting an action of an automorphism of order six
- 146 designs

Known triplanes

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 5421
71	15	3	≥ 146
81	16	3	?

Motivation

OPEN PROBLEMS

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 5421
71	15	3	≥ 146
81	16	3	?

+ designs admitting only the trivial automorphism

automorphisms of prime order?
the trivial automorphism?

?

The smallest triplane whose existence is not known, i.e. a triplane with parameters $(81, 16, 3)$, is for many years also the smallest symmetric design for which (non)existence is not determined (since 1985 when the existence of a symmetric 2-$(78, 22, 6)$ design was established[2]).

[2] Z. Janko, T. van Trung, Construction of a new symmetric block design for $(78, 22, 6)$ with the help of tactical decompositions, J. Combin. Theory A 40 (1985) 451–455.

Known triplanes of order twelve

The full automorphism groups of known triplanes of order twelve

the order of G	the structure of G	the number of designs
336	$(E_8 : F_{21}) \times Z_2$	6
168	$E_8 : F_{21}$	3
48	$E_4 \times A_4$	26
42	$F_{21} \times Z_2$	6
24	$A_4 \times Z_2$	89
24	$S_3 \times E_4$	16

Known actions:

- $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
- $f_3 \in \{2, 5\}$
- $f_7 = 1$.

Automorphisms of order 5, 11 and 13

Lemma (order 5)

Let $\mathcal{D}$ be a triplane with parameters $(71, 15, 3)$ and let ρ be an automorphism of $\mathcal{D}$. If $|\rho| = 5$, then ρ fixes exactly one point (block) of $\mathcal{D}$.

Lemma (order 13)

Let $\mathcal{D}$ be a triplane with parameters $(71, 15, 3)$ and let ρ be an automorphism of $\mathcal{D}$. Then the order of ρ is not equal to 13.

Lemma (order 11)

Let $\mathcal{D}$ be a triplane with parameters $(71, 15, 3)$ and let ρ, $|\rho| = 11$, be an automorphism of $\mathcal{D}$. Then ρ acts on $\mathcal{D}$ with five fixed points (blocks) and the corresponding orbit matrix OM.

OM	1	1	1	1	1	11	11	11	11	11	11
1	1	1	1	1	0	11	0	0	0	0	0
1	1	1	1	0	1	0	11	0	0	0	0
1	1	1	0	1	1	0	0	11	0	0	0
1	1	0	1	1	1	0	0	0	11	0	0
1	0	1	1	1	1	0	0	0	0	11	0
11	1	0	0	0	0	2	2	2	2	3	3
11	0	1	0	0	0	2	2	2	3	2	3
11	0	0	1	0	0	2	2	3	2	2	3
11	0	0	0	1	0	2	3	2	2	2	3
11	0	0	0	0	1	3	2	2	2	2	3
11	0	0	0	0	0	3	3	3	3	3	0

Automorphisms of triplanes of order twelve

Theorem

Let $\mathcal{D}$ be a triplane of order twelve and let ρ be an automorphism of prime order acting on $\mathcal{D}$. Let f_i denotes the number of fixed point of ρ, where $|\rho| = i$. Then $|\rho| \in \{2, 3, 5, 7, 11\}$ and

1. $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
2. $f_3 \in \{2, 5\}$,
3. $f_5 = f_7 = 1$,
4. $f_{11} = 5$.

Codes

A **code** C of length n over the alphabet Q is a subset $C \subseteq Q^n$. Elements of a code are called **codewords**. A code C is called a p-ary **linear code** of dimension m if $Q = \mathbb{F}_p$, for a prime p, and C is an m-dimensional subspace of a vector space $\mathbb{F}_p^n$.

The **code** $C_{\mathbb{F}}$ **of the design** $\mathcal{D}$ over the finite field $\mathbb{F}$ is the space spanned by the incidence vectors of the blocks over $\mathbb{F}$. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_{\mathbb{F}} = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $\mathbb{F}^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to $\mathbb{F}$.

Construction of new triplanes

The number of vertices of $\mathcal{G}_i$, $i \in \{1, 2, ..., 146\}$, varies from 120 to 59648.

From the code $\mathcal{C}_{30}$ we obtained three isomorphic copies of a design with the full automorphism group of order 8, which will be denoted by $\mathcal{D}$. The dual of $\mathcal{D}$ is obtained from the code $\mathcal{C}_{33}$ spanned by $\mathcal{D}_{33}$. Those two designs are the first examples of a triplane of order twelve that do not admit an action of an automorphism of order three.

i	cw15	N	i	cw15	N	i	cw15	N	i	cw15	N	i	cw15	N
1	1148	96	31	228	16	61	386	1	91	647	1	121	1262	1
2	1844	2	32	958	1	62	212	1	92	2736	16	122	926	1
3	488	2	33!	6554	1	63	338	1	93	2448	16	123	4814	2
4	368	2	34	1278	1	64	272	1	94	4342	1	124*	18364	256
5	128	1	35	986	1	65!	6226	8	95	3888	16	125	2876	2
6	338	1	36	774	1	66	1762	4	96	1154	1	126*	15788	256
7	282	1	37	758	1	67!	6186	8	97	3974	1	127	4778	2
8	192	1	38	1278	1	68	1816	4	98	3168	8	128	5594	2
9	366	1	39	1394	2	69!	6226	8	99	3388	8	129*	14132	256
10	236	1	40*	9434	8	70	1714	4	100	3678	1	130	2796	2
11	338	1	41	3856	16	71!	6186	8	101*	19788	256	131	1884	2
12	248	1	42*	8632	1	72	1966	4	102	1258	1	132	2788	2
13	178	1	43*	26538	128	73	1115	1	103*	28132	256	133	5884	16
14	314	1	44	3588	16	74	844	1	104*	17788	256	134	3936	16
15	366	1	45	2888	8	75	161	1	105	3868	8	135*	28276	256
16	266	1	46	1126	1	76	2688	2	106*	21212	512	136	2856	2
17!	6454	8	47	1482	1	77	898	1	107*	14957	2	137*	23788	512
18	2868	4	48	614	1	78	1948	2	108*	29936	3872	138	3288	2
19!	6558	8	49	1142	2	79	1948	2	109	4144	16	139	4366	1
20	2224	4	50	324	2	80	946	1	110*	18668	512	140*	14828	256
21!	6454	8	51	348	2	81	2688	2	111*	21212	512	141	2788	2
22	1762	4	52	1228	2	82	378	1	112	3698	16	142	3888	8
23!	6558	8	53	178	1	83	7388	32	113	3794	8	143	3472	8
24	2374	4	54	266	1	84*	59648	32768	114	3788	8	144	43338	4896
25	1126	1	55	378	1	85	3868	8	115	5174	1	145	5486	8
26	2988	8	56	386	1	86	3452	8	116*	18244	256	146	718	1
27	4548	16	57	434	1	87*	59648	32768	117*	22688	512			
28	1818	1	58	344	1	88	7932	32	118*	11484	1			
29*	26538	896	59	224	1	89*	24758	4896	119*	28952	256			
30!	5752	1	60	314	1	90	3658	8	120	2948	2			

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THANK YOU FOR LISTENING!