

A few new triplanes

Motivation -
known results

Automorphisms
of triplanes of
order twelve

A construction of
new triplanes of
order twelve

A few new triplanes

(joint work with Dean Crnković)

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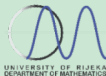
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An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set \mathcal{P} , block set \mathcal{B} and incidence I is a t - (v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. We consider triplanes, i.e. symmetric block designs with $\lambda = 3$. Triplanes of order 12, i.e. symmetric $(71, 15, 3)$ designs, have the greatest number of points among all known triplanes and it is not known if a triplane $(v, k, 3)$ exists for $v > 71$.

In this talk, in addition to reviewing previously known results, we give the first example of a triplane of order 12 that doesn't admit an automorphism of order 3.



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- ① Motivation - known results
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Known triplanes for $v \leq 100$ and $k \leq \frac{v}{2}$ ¹

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 3752
71	15	3	≥ 72
81	16	3	?

P.B.Gibbons, *Computing Techniques for the Construction and Analysis of Block Designs*, PhD Thesis, University of Toronto, 1976.

R.H.F.Denniston, *Enumeration of symmetric designs (25,9,3)*, *Ann. Discrete Math.* 15 (1982), 111-127.

E. Spence, *A complete classification of symmetric (31,10,3) designs*, *Des. Codes Crypt.* 2 (1992), 127-136.

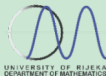
R.A.Mathon, E. Spence, *On 2-(45,12,3) designs*, *J. Combin. Des.* 4 (1996), 155-175.

W.H.Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Mathematisch Centrum, Amsterdam, 1980.

S. Rukavina, *Some new triplanes of order twelve*, *Glas. Mat. Ser. III* 36(56) (2001), 105-125

¹R. Mathon, A.Rosa, *2-(v, k, λ) Designs of Small Order*, in: *Handbook of Combinatorial Designs*, 2nd ed. (C. J. Colbourn and J. H. Dinitz, Eds.), Chapman and Hall/CRC, Boca Raton, 2007, 25–57.

Triplanes of order nine - CRC, 2007



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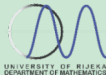
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- R. A. Mathon and E. Spence, On 2-(45, 12, 3) designs, J. Combin. Des. 4 (1996), 155–175
 - R. M. MacFarland, A family of difference sets in non-cyclic groups, J. Comb. Th., A 15 (1973) 1–10.
 - T. Kölmel, Einbettbarkeit symmetrischer (45, 12, 3) Blockpläne mit fixpunktfrei operierenden Automorphismen, Heidelberg 1991.
 - V. Čepulić, On symmetric block designs (45, 12, 3) with automorphism of order 5, Ars Combin., 37 (1994) 37-48.

- 1136 designs with trivial automorphism, under the assumption that the incidence matrix of a 2-(45, 12, 3) design has a certain block structure

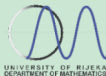
- all 2-(45, 12, 3) designs having an automorphism of order 5 or 11
- the total number of non-isomorphic 2-(45, 12, 3) designs found is 3752



- D. Crnković, D. Dumičić Danilović, SR, Enumeration of symmetric $(45, 12, 3)$ designs with nontrivial automorphisms, J. Algebra Comb. Discrete Struct. Appl. 3 (2016), 145–154.
 - U. Dempwolff, Primitive rank 3 groups on symmetric designs, Des. Codes Cryptogr. 22(2) (2001), 191–207.
 - D. Crnković, SR, M. Schmidt, A Classification of all Symmetric Block Designs of Order Nine with an Automorphism of Order Six, J. Combin. Des. 14, No.4 (2006), 301–312.
 - K. Coolsaet, J. Degraer, E. Spence, The strongly regular $(45, 12, 3, 3)$ graphs, Electron. J. Combin. 13 (2006), no. 1, Research Paper 32, 9 pp.

- There are exactly 4285 symmetric $(45, 12, 3)$ designs that admit nontrivial automorphisms. Among them there are 1161 self-dual designs and 1562 pairs of mutually dual designs.

Triplanes of order twelve - CRC, 2007



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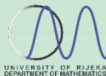
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- W. H. Haemers, Eigenvalue Techniques in Design and Graph Theory, Mathematisch Centrum, Amsterdam, 1980.
- SR, Some new triplanes of order twelve, Glas. Mat. Ser. III 36(56) (2001), 105–125.
 - M. Garapić, Construction of some new triplanes. Ph. D. Thesis, University of Zagreb, 1993.

- The first triplane of order twelve has been constructed from an embeddable 2 - $(56, 12, 3)$ design. Haemers constructed four mutually non-isomorphic symmetric $(71, 15, 3)$ designs which are not self-dual. Three of the constructed triplanes have the full automorphism group of order 336 and the order of the full automorphism group of the fourth triplane is 48.

- 72 designs with an automorphism of order six acting with one fixed point



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- D. Crnković, SR, L. Simčić, On triplanes of order twelve admitting an automorphism of order six and their binary and ternary codes, Util. Math. 103 (2017), 23–40.
 - SR, 2-(56,12,3) designs and their class graphs, Glas. Mat. Ser. III 38(58) (2003), 201-210.

- all triplanes of order twelve admitting an action of an automorphism of order six
- 146 designs

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 5421
71	15	3	≥ 146
81	16	3	?



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OPEN PROBLEMS

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 5421
71	15	3	≥ 146
81	16	3	?

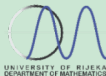
+ designs admitting only the trivial automorphism

automorphisms of prime order?
the trivial automorphism?

The smallest triplane whose existence is not known, i.e. a triplane with parameters $(81, 16, 3)$, is for many years also the smallest symmetric design for which (non)existence is not determined (since 1985 when the existence of a symmetric 2- $(78, 22, 6)$ design was established²).

²Z. Janko, T. van Trung, Construction of a new symmetric block design for $(78, 22, 6)$ with the help of tactical decompositions, J. Combin. Theory A 40 (1985) 451–455.

Known triplanes of order twelve



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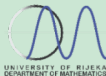
The full automorphism groups of known triplanes of order twelve

the order of G	the structure of G	the number of designs
336	$(E_8 : F_{21}) \times Z_2$	6
168	$E_8 : F_{21}$	3
48	$E_4 \times A_4$	26
42	$F_{21} \times Z_2$	6
24	$A_4 \times Z_2$	89
24	$S_3 \times E_4$	16

Known actions:

- $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
- $f_3 \in \{2, 5\}$
- $f_7 = 1$.

Automorphisms of order 5, 11 and 13



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Lemma (order 5)

Let \mathcal{D} be a triplane with parameters $(71, 15, 3)$ and let ρ be an automorphism of \mathcal{D} . If $|\rho| = 5$, then ρ fixes exactly one point (block) of \mathcal{D} .

Lemma (order 13)

Let \mathcal{D} be a triplane with parameters $(71, 15, 3)$ and let ρ be an automorphism of \mathcal{D} . Then the order of ρ is not equal to 13.

Lemma (order 11)

Let \mathcal{D} be a triplane with parameters $(71, 15, 3)$ and let ρ , $|\rho| = 11$, be an automorphism of \mathcal{D} . Then ρ acts on \mathcal{D} with five fixed points (blocks) and the corresponding orbit matrix OM .

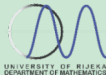
OM	1	1	1	1	1	11	11	11	11	11	11
1	1	1	1	1	0	11	0	0	0	0	0
1	1	1	1	0	1	0	11	0	0	0	0
1	1	1	0	1	1	0	0	11	0	0	0
1	1	0	1	1	1	0	0	0	11	0	0
1	0	1	1	1	1	0	0	0	0	11	0
11	1	0	0	0	0	2	2	2	2	3	3
11	0	1	0	0	0	2	2	2	3	2	3
11	0	0	1	0	0	2	2	3	2	2	3
11	0	0	0	1	0	2	3	2	2	2	3
11	0	0	0	0	1	3	2	2	2	2	3
11	0	0	0	0	0	3	3	3	3	3	0

Automorphisms of triplanes of order twelve

Theorem

Let \mathcal{D} be a triplane of order twelve and let ρ be an automorphism of prime order acting on \mathcal{D} . Let f_i denotes the number of fixed point of ρ , where $|\rho| = i$. Then $|\rho| \in \{2, 3, 5, 7, 11\}$ and

- 1 $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
- 2 $f_3 \in \{2, 5\}$,
- 3 $f_5 = f_7 = 1$,
- 4 $f_{11} = 5$.



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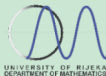
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A **code** C of length n over the alphabet Q is a subset $C \subseteq Q^n$. Elements of a code are called **codewords**. A code C is called a p -ary **linear code** of dimension m if $Q = \mathbb{F}_p$, for a prime p , and C is an m -dimensional subspace of a vector space \mathbb{F}_p^n .

The **code** $C_{\mathbb{F}}$ of the design \mathcal{D} over the finite field \mathbb{F} is the space spanned by the incidence vectors of the blocks over \mathbb{F} . If the point set of \mathcal{D} is denoted by \mathcal{P} and the block set by \mathcal{B} , and if Q is any subset of \mathcal{P} , then we will denote the incidence vector of Q by v^Q . Thus $C_{\mathbb{F}} = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $\mathbb{F}^{\mathcal{P}}$, the full vector space of functions from \mathcal{P} to \mathbb{F} .

Construction of new triplanes



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The number of vertices of \mathcal{G}_i , $i \in \{1, 2, \dots, 146\}$, varies from 120 to 59648.

From the code \mathcal{C}_{30} we obtained three isomorphic copies of a design with the full automorphism group of order 8, which will be denoted by \mathcal{D} . The dual of \mathcal{D} is obtained from the code \mathcal{C}_{33} spanned by \mathcal{D}_{33} . Those two designs are the first examples of a triplane of order twelve that do not admit an action of an automorphism of order three.

i	code	$ \mathcal{G}_i $	$ \mathcal{C}_i $	$ \mathcal{D}_i $	$ \mathcal{C}_i $	$ \mathcal{D}_i $	$ \mathcal{C}_i $	$ \mathcal{D}_i $	$ \mathcal{C}_i $	$ \mathcal{D}_i $				
1	1048	84	3,3	228	38	8,1	388	5	9,1	847	5	12,1	1282	5
2	8844	2	3,2	938	3	6,2	232	3	9,2	2738	38	122	938	3
3	188	2	3,2	751,2	1,2	6,1	339	3	9,1	2438	38	122	938,2	2
4	188	2	3,4	127,6	3	8,4	37,2	3	9,4	432	3	12,4	1838,1	3,1,1
5	128	3	3,3	886	3	6,3	822,1	8	9,3	1838	8	12,3	187,8	8
6	336	3	3,6	774	3	6,6	128,2	4	8,6	1034	5	12,6	1874,1	5,1,1
7	888	3	3,7	738	3	7,7	938,1	3	8,7	187,8	3	12,7	177,8	3
8	188	3	3,8	127,6	3	8,8	181,6	1	9,8	3388	8	12,8	138,1	2
9	336	3	3,8	128,4	2	7,8	432,1	8	8,8	3388	8	12,8	183,1	2,1,1
10	336	3	3,8	128,4	2	7,8	173,4	4	8,8	187,8	3	12,8	278,1	2
11	336	3	3,8	128,4	2	7,8	181,6	1	8,8	187,8	3	12,8	188,1	2
12	248	3	3,8	93,2	3	7,8	188,6	4	8,8	187,8	3	12,8	278,1	2
13	178	3	3,8	128,4	2	7,8	133,3	3	8,8	187,8	3	12,8	188,1	2
14	336	3	3,8	128,4	2	7,8	84,1	1	8,8	187,8	3	12,8	188,1	2
15	336	3	3,8	128,4	2	7,8	181,6	1	8,8	187,8	3	12,8	188,1	2
16	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
17	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
18	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
19	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
20	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
21	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
22	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
23	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
24	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
25	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
26	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
27	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
28	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
29	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
30	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
31	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
32	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
33	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
34	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
35	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
36	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
37	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
38	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
39	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
40	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
41	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
42	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
43	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
44	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
45	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
46	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
47	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
48	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
49	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
50	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
51	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
52	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
53	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
54	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
55	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
56	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
57	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
58	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
59	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
60	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
61	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
62	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
63	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
64	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
65	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
66	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
67	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
68	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
69	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
70	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
71	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
72	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
73	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
74	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
75	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
76	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
77	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
78	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
79	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
80	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
81	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
82	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
83	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
84	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
85	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
86	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
87	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
88	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
89	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
90	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
91	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
92	888	3	3,8	128,4	2	7,8	288,8	2	8,8	232,2	3,1,2	12,8	288,1	2
93	888	3	3,8	128,4	2									

A few new triplanes

Motivation - known results

Automorphisms of triplanes of order twelve

A construction of new triplanes of order twelve

OPEN PROBLEMS

v	k	λ	the number of designs
15	7	3	5
25	9	3	78
31	10	3	151
45	12	3	≥ 5421
71	15	3	≥ 148
81	16	3	?

+ designs admitting only the trivial automorphism

automorphisms of prime order?
the trivial automorphism?

$$|\rho| \in \{2, 3, 5, 7, 11\}$$

$$f_2 \in \{7, 9, 11, 13, 15, 17\},$$

$$f_3 \in \{2, 5\},$$

$$f_5 = f_7 = 1,$$

$$f_{11} = 5.$$

THANK YOU FOR LISTENING!