Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations

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How to extend an holomorphic object to a neighborhood ?

C^d is a compact complex d-manifold embedded (holomorphically) in complex manifolds M^{n+d}

$$C \hookrightarrow M \rightsquigarrow \text{Normal bundle } N_C := TM/TC.$$

Idea :

- Transform a neighborhood of C in M into a neighborhood of C (the 0th section) in N_C.
- Extend to a neighborhood of 0th in N_C
- Pull-back the extension to a neighborhood of C in M

Embedding of compact complex submanifolds.

C^d is a compact complex d-manifold embedded (holomorpically) in complex manifolds M_1^{n+d}, M_2^{n+d}

$$C^d \hookrightarrow \begin{cases} M_1^{n+d} \\ M_2^{n+d} \end{cases}$$

Take a (arbitrary small) neighborhood U_i of C in M_i

Question : Is U_1 holomorphically equivalent to U_2 ?

$\rightarrow$ Grauert's "Formale Prinzip" question : If "U_1 is **formally equivalent** to U_2", is U_1 **holomorphically** equivalent to U_2 ?

Some answers

→ If N_C is *negative*, then Grauert (Hironaka-Rossi) showed

formal equivalence ⇒ holomorphic equivalence

→ If N_C is *positive*, then Griffiths showed

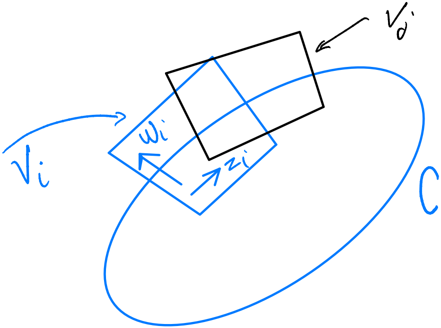
equivalence up to some finite order ⇒ holomorphic equivalence

→ Arnold'construction : **elliptic curve** $\mathbb{T}^1 \hookrightarrow M^2$ s.t. neighborhood of $\mathbb{T}^1$ **formally equiv.** to neighborhood of 0th section in $N_{\mathbb{T}^1}$ and **holomorphic equiv.** based on *small divisors condition*.

→ Ilyashenko-Pyartli : ext. of Arnold's construction to **direct product** of elliptic curves the normal bdle of which is **direct sum of line bdles** ←-- *small divisors condition*.

Coordinates Patch

Covering $\mathcal{U} = \{U_i\}$ of C with coordinates z_i.
Covering $\mathcal{V} = \{V_i\}$ of a neighborhood of C in M : $U_i = V_i \cap C$ with coordinates (z_i, w_i), $U_i := \{w_i = 0\}$.



$U_i \cap U_j$

$\{ z_i = \varphi_{ij}(z_j) \}$

transition fct of C

Neighborhoods in coordinates

"Split"

Assumption : $TM_{|C} = TC \oplus N_C$.
In coordinates, the transitions functions on $V_k \cap V_j$

- of the normal bundle $N_C(M)$ is $N_{kj}(z_j, w_j)$:

$$\begin{aligned} w_k &= t_{kj}(z_j)w_j \\ z_k &= \phi_{kj}(z_j). \end{aligned}$$

- of the neighborhood of the embedding is $\Phi_{kj} := N_{kj}(z_j, w_j) + \phi_{kj}^{\bullet}$

$$\begin{aligned} w_k &= t_{kj}(z_j)w_j + \sum_{Q\in\mathbb{N}^d, |Q|\geq 2} \phi_{kj,Q}^{v}(z_j)w_j^Q \\ z_k &= \phi_{kj}(z_j) + \sum_{Q\in\mathbb{N}^d, |Q|\geq 2} \phi_{kj,Q}^{h}(z_j)w_j^Q. \end{aligned}$$

Holomorphic (resp. formal) Equivalence Problem

To find $F_j = \mathrm{I} + f_j$ holomorphic (resp. formal) on V_j such that

$$F_k \Phi_{kj} = N_{kj} F_j$$

$$\begin{aligned} w_j &= v_j + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} f^v(h_j) v_j^Q \\ z_j &= h_j + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} f^h(h_j) v_j^Q. \end{aligned}$$

Computations

To simplify exposition: N_C **flat** $\rightsquigarrow t_{kj} = cst.$

$$F_j = \mathrm{I} + (f_j^h, f_j^v), \quad F_k \Phi_{kj} = N_{kj} F_j$$

We obtain the horizontal equation

$$\begin{aligned}
\phi_{kj}(h_j) + \phi_{kj}^h + f_k^h(\phi_{kj} + \phi_{kj}^h, t_{kj} v_j + \phi_{kj}^v) &= \phi_{kj}(h_j + f_j^h(h_j, v_j)) \\
&\quad + \phi_{kj}^h(h_j + f_j^h, v_j + f_j^v). \\
f_k^h(\phi_{kj}(h_j), t_{kj} v_j) - D\phi_{kj}(h_j) f_j^h(h_j, v_j) &= -\phi_{kj}^h(h_j, v_j) \\
&\quad + \text{nonlinear in the unknowns} \\
&= R_{kj}^h
\end{aligned}$$

The vertical equation reads

$$\begin{aligned}
f_k^v(\phi_{kj}(h_j), t_{kj} v_j) - t_{kj} f_j^v(h_j, v_j) &= -\phi_{kj}^v(h_j, v_j) \\
&\quad + \text{nonlinear in the unknowns} \\
&= R_{kj}^v
\end{aligned}$$

Cohomology operators

Idea: step by step on homogenous degree $m \geq 2$ wrt to v_j :

$$\begin{aligned}\delta[f]_m &= \begin{pmatrix} \delta^h(\{[f_j^h]_m\}) \\ \delta^v(\{[f_j^v]_m\}) \end{pmatrix} = \{[R_{kj}]_m\} \\ &= \mathcal{F}_m([f]_2, \ldots, [f]_{m-1}, [\phi^\bullet]_2, \ldots, [\phi^\bullet]_m).\end{aligned}$$

$$\begin{aligned}\delta^h(\{[f_j^h]_m\}) &= \{\left[R_{kj}^h\right]_m\} \in C^1\left(\mathcal{U}, TM_{|C} \otimes S^m(N_C^*)\right) \\ \delta^v(\{[f_j^v]_m\}) &= \{\left[R_{kj}^v\right]_m\} \in C^1\left(\mathcal{U}, N_C \otimes S^m(N_C^*)\right)\end{aligned}$$

$$C^0(\mathcal{U}, E) \xrightarrow{\delta} C^1(\mathcal{U}, E) \xrightarrow{\delta} C^2(\mathcal{U}, E) \xrightarrow{\delta} \cdots$$

→ Need to **solve cohomological equations**
→ Need to **estimate** solutions → **“small divisors”**

Case of torus

In torus case, once can manage jus to have 1 equation with 1 unknown

$$f(\omega + h, \lambda_1 v_1, \cdots, \lambda_n v_n) - \mathrm{diag}(\lambda_1, \ldots, \lambda_n) f(h, z) = \cdots$$

$$\rightsquigarrow \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{Q\in\mathbb{N}^n, |Q|\geq 2} \left(e^{2\pi i \mathbf{k}.\omega} \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_i\right) f_{i,\mathbf{k},Q} e^{2\pi i \mathbf{k}.h} v^Q = \cdots$$

Estimates of solutions the cohomological equations

$\rightarrow$ I.F. Donin : $\{\mathcal{U}^\epsilon\}$ family of covering of C: $f \in C^1(U^\epsilon, E)$; $[f] = 0$ in H^1. then : $\exists u \in C^0(U^{\epsilon/3}, E)$ s.t. $\delta u = f$ and $\|u\|_{\epsilon/3} \leq \frac{K}{\epsilon^\tau}\|f\|_\epsilon$

Proposition ("pumping")

$\exists$ $\{\mathcal{U}^r\}_{r_* \leq r \leq r^*}$ *family of "polydiscs" coverings of* C *s.t.*

$$\|u\|_{\mathbf{r}} \leq K(E)\|f\|_{\mathbf{r}}, \quad \delta u = f$$

Our "full linearizaton" result (flat case)

Let $\eta_0 := 1$ and $d_m := \max\left(K(N_C \otimes S^m(N_C^*)), K(T_C \otimes S^m(N_C^*))\right)$

$$\eta_m := d_m \max_{m_1+\cdots+m_p+s=m} \eta_{m_1} \cdots \eta_{m_p},$$

Theorem (Linearization of neighborhoods)

Assumptions :

- $(M, C) \widehat{\equiv} (N_C, C)$
- N_C *flat and unitary*
- $\eta_m \leq L^m$
- $H^0(C, T_C M \otimes S^\ell(N_C^*)) = 0$, *for all* $\ell > 1$

Conclusion $(M, C) \equiv (N_C, C)$

Existence of foliation with leaf C

If there is a neighborhood s.t.

$$\begin{aligned} w_k &= t_{kj} w_j \\ z_k &= \phi_{kj}(z_j) + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \tilde{\phi}^h_{kj,Q}(z_j) w_j^Q. \end{aligned}$$

then $w_j = cst$ gives a foliation.

$\rightsquigarrow$ "vertical" linearization.

One result

Let $\eta_0 := 1$ and $d_m := K(N_C \otimes S^m(N_C^*))$

$$\eta_m := d_m \max_{m_1+\cdots+m_p+s=m} \eta_{m_1} \cdots \eta_{m_p},$$

Theorem (Vertical linearization of neighborhoods)

Assumptions :

- (M, C) *is "formally vertically linearizable"*
- N_C *flat and unitary*
- $\eta_m \leq L^m$
- $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$, *for all* $\ell > 1$

Conclusion (M, C) *is "holomorphically vertically linearizable"*

$\rightarrow$ Ueda : complex curve in a surface ($n = d = 1$)