QUASIBANDS AND NONCOMMUTATIVE, NONASSOCIATIVE LATTICES

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Joint work with T. Pisanski

Motivation: regular bands

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A band is *regular* (not to be confused with a myriad of other uses of the same word) if each of Green's equivalence relations $\mathcal{L}$ and $\mathcal{R}$ are congruences. Here

$$x \ \mathcal{L} \ y \iff xy = x \text{ and } yx = y$$
$$x \ \mathcal{R} \ y \iff xy = y \text{ and } yx = x$$

It turns out that a band is regular if and only if it satisfies the following identity:

$$xyxzx = xyzx \,.$$

Motivation II

In a band B, define a new binary operation $\circ$ by $x \circ y = xyx$.

Theorem (MK 2005, unpublished)

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Idea of proof.

Expand both sides of the associative law for $\circ$:

$$(x \circ y) \circ z = xyxzxyx$$
$$x \circ (y \circ z) = xyzyx.$$

Thus $\circ$ is associative if and only if

$$xyzyx = xyxzxyx.$$

Then just check by about a dozen steps of calculation that this last identity is equivalent to the more familiar identity $xyxzx = xyzx$. $\qquad\square$

Motivation III

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Surprisingly, a (nearly) complete story emerged. We will start by taking some of the identities satisfied by $\circ$ and using them to define a new algebraic structure.

Quasibands

A (left) *quasiband* $(B, \circ)$ is a magma satisfying the following identities.

$$x \circ x = x \tag{M1}$$

$$x \circ (y \circ x) = x \circ y \tag{M2}$$

$$x \circ (y \circ (x \circ z)) = (x \circ y) \circ z \tag{M3}$$

$$(x \circ (y \circ z)) \circ ((z \circ (y \circ x)) \circ u) = x \circ (y \circ (z \circ u)) \tag{M4}$$

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These identities are independent of each other. The fourth one probably makes your eyes glaze over, but it does have some meaning I will discuss later.

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If B is a band and $x \circ y = xyx$, then $(B, \circ)$ is a quasiband, called the *induced quasiband* of B.

Counting

Associative quasibands are precisely the same as *left regular* bands, satisfying $xyx = xy$.

How these things grow, up to isomorphism:

Order	1	2	3	4	5	6
#(left reg bands)	1	2	6	23	109	630
#(quasibands)	1	2	6	23	110	642

Example

Here is a 7-element band and its induced quasiband.

·	0	1	2	3	4	5	6
0	0	6	4	0	4	5	6
1	3	1	5	3	4	5	1
2	2	2	2	2	2	2	2
3	3	1	4	3	4	5	1
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	0	6	5	0	4	5	6

∘	0	1	2	3	4	5	6
0	0	0	4	0	4	5	0
1	1	1	5	1	4	5	1
2	2	2	2	2	2	2	2
3	3	3	4	3	4	5	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	5	6	4	5	6

The subset $A = \{0, 1, 2, 3, 4, 5\}$ is a subquasiband but not a subband. $(A, \circ)$ is not associative since

$$(0 \circ 1) \circ 2 = 0 \circ 2 = 4 \neq 5 = 0 \circ 5 = 0 \circ (1 \circ 2).$$

A question

At this point, you're probably wondering:

How do we know this is the "right" definition? Surely induced quasibands satisfy many identities. Why are the four identities in the definition of quasiband enough?

Good question! And it deserves a complete answer. Which I defer until after ...

Natural preorder and partial order

On a quasiband $(B, \circ)$, define a relation:

$$x \preceq y \quad \Longleftrightarrow \quad x \circ y = x.$$

Theorem

$\preceq$ is a preorder and is compatible with $\circ$, that is, $x \preceq y$ implies $x \circ z \preceq y \circ z$ and $z \circ x \preceq z \circ y$.

Let $\mathcal{D}$ be the equivalence relation induced by $\preceq$, that is, $x \mathrel{\mathcal{D}} y$ if and only if $x \preceq y$ and $y \preceq x$.

Theorem

$\mathcal{D}$ is the smallest semilattice congruence on B.

If $\circ$ is associative, $\mathcal{D}$ coincides with the usual Green's relation.

Natural partial order

On a quasiband $(B, \circ)$, define a relation:

$$x \leq y \quad \Longleftrightarrow \quad y \circ x = x.$$

Theorem

$\leq$ is a partial order that refines $\preceq$, that is, if $x \leq y$ then $x \preceq y$.

If $\circ$ is associative, $\leq$ coincides with the usual natural partial order.

Multiplication semigroups

For each a in a quasiband $(B, \circ)$, define

$$M_a : B \to B; x \mapsto a \circ x .$$

Let $\mathrm{Mlt}(X) = \langle M_a \mid a \in B \rangle$ be the transformation semigroup (called the *left multiplication semigroup*) generated by the M_a's.

Theorem

Let $(B, \circ)$ be a quasiband. Then:

1. $\mathrm{Mlt}(B) = \{M_a M_b \mid a, b \in B\}$*;*
2. $\mathrm{Mlt}(B)$ *is a band;*
3. *The set $M(B) = \{M_a \mid a \in B\}$ is a subquasiband of $(\mathrm{Mlt}(B), \circ)$;*
4. *the mapping $\mu : B \to M(B); a \mapsto M_a$ is an isomorphism of quasibands.*

Sketch of Proof

(1) Remember the nasty identity (M4)? (Of course not.) Here it is in terms of the multiplication maps:

$$M_x M_y M_z = M_{x \circ (y \circ z)} M_{z \circ (y \circ x)}$$

It says that any composition of three M's can be reduced to a composition of two M's.

By an easy induction, any composition of M's can be reduced to just two.

(2) follows (1) and another identity that follows from the axioms: $M_x M_y M_x M_y = M_x M_y$.

(3) is essentially just (M3): $M_x \circ M_y = M_x M_y M_x = M_{x \circ y}$.

(4): the hard part is showing the map is injective. This uses the natural partial order $\leq$.

Upshot

> **Corollary**
>
> *Every quasiband is isomorphic to a subquasiband of an induced quasiband.*
>
> *Thus quasibands are precisely the ∘-subreducts of bands.*

In other words: our axioms were enough! Induced quasibands do not satisfy any identities not satisfied by all quasibands.

Remark

The assignment $B \rightsquigarrow \mathrm{MIt}(B)$ is not functorial. There is a different approach which goes as follows.

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For a quasiband $(B, \circ)$, let F_B denote the free band on the set B, and let

$$\mathcal{U}_B = F_B / \equiv$$

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where $\equiv$ is the congruence on F_B generated by the relation $xyx = x \circ y$.

Then $\mathcal{U}_B$ is another band which contains B as a subquasiband. This time the assignment $B \rightsquigarrow \mathcal{U}_B$ is functorial.

Problem

If FQ_X is the free quasiband on a set X, what is the relationship between $\mathrm{Mlt}(FQ_X)$ and F_X?

Noncommutative, nonassociative lattices

Let $(S, \wedge, \vee)$ be a double band, that is, $(S, \wedge)$ and $(S, \vee)$ are bands on the same underlying set S. We use $\curlywedge$ to denote the induced left quasiband operation of $(S, \wedge)$ and $\curlyvee$ to denote the induced *right* quasiband operation of $(S, \vee)$. Thus

$$x \curlywedge y = x \wedge y \wedge x\,,$$
$$x \curlyvee y = y \vee x \vee y$$

(The convention of using the induced left quasiband for $\wedge$ and the induced right quasiband for $\vee$ is chosen to match some already existing conventions in the noncommutative lattice literature.)

Double quasibands

Thus our *double quasiband* $(S, \curlywedge, \curlyvee)$ satisfies the following axioms:

$$x \curlywedge x = x \qquad\qquad x \curlyvee x = x$$

$$x \curlywedge (y \curlywedge x) = x \curlywedge y \qquad\qquad (x \curlyvee y) \curlyvee x = y \curlyvee x$$

$$x \curlywedge (y \curlywedge (x \curlywedge z)) = (x \curlywedge y) \curlywedge z \qquad ((z \curlyvee x) \curlyvee y) \curlyvee x = z \curlyvee (y \curlyvee x)$$

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What about absorption laws?

Absorption

Well-motivated absorption laws express duality between the natural preorders or orders of each operation. So take your pick!

Weakest reasonable notion:

$$x \leq_\curlywedge y \implies y \preceq_\curlyvee x \qquad\qquad (x \curlywedge y) \curlyvee x = x$$
$$x \leq_\curlyvee y \implies y \preceq_\curlywedge x \qquad\qquad x \curlywedge (y \curlyvee x) = x$$

Analogous to Leech's quasilattices:

$$x \preceq_\curlywedge y \implies y \preceq_\curlyvee x \qquad\qquad (y \curlywedge x) \curlyvee x = x$$
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Analogous to Leech's paralattices:

$$x \leq_\curlywedge y \implies y \leq_\curlyvee x \qquad\qquad x \curlyvee (x \curlywedge y) = x$$
$$x \leq_\curlyvee y \implies y \leq_\curlywedge x \qquad\qquad (y \curlyvee x) \curlywedge x = x$$

Remarks

Many commonly studied classes of noncommutative lattices (e.g., skew lattices, antilattices) have regular band reducts. This means that in these classes, the induced double quasibands are actually noncommutative (associative!) lattices in their own right.

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For example, if $(S, \wedge, \vee)$ is a refined quasilattice (e.g., a skew lattice), then $(S, \curlywedge, \curlyvee)$ is a left-handed skew lattice.

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So to really get a sense of what is happening in the nonassociative case, we need more "natural" (whatever that means) examples of noncommutative lattices with nonregular band reducts. This will give us "natural" examples of nonassociative induced double quasibands to play with.

Thanks!