Very weak solutions to PDEs in inhomogeneous and anisotropic spaces

European Women in Mathematics

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What is definition of solution that makes sense?

Who can be called ‘a solution’ for general μ?

A function $u \in W^1 L_M(\Omega)$ is called a weak solution to a problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\text{if} \quad \int_\Omega \mathcal{A}(x, Du) \cdot D\phi \, dx = \int_\Omega \phi \, d\mu(x) \quad \text{for every } \phi \in C_c^\infty(\Omega).$$

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Weak solutions are too restrictive,
distributional solutions can be not unique & wild... :(

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Already for $-\Delta_p u = \delta_0$, $1 < p < n$, in $B(0,1)$
we deal with the so-called fundamental solution

$$G(x) = c_{n,p}\left(|x|^{\frac{p-n}{p-1}} - 1\right) \quad \text{if } |x| \neq 0,$$

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Most ideas go through some approximate procedure $-\mathrm{div}\mathcal{A}(x, Du_s) = f_s$ or restricting the family of test functions or energy of solutions.

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- not reflexive – there is no proper definition of $(L_M)'$, density of smooth functions is possible only in the modular topology (it's due to general growth of M),
- inhomogeneous (so smooth functions are not necessarily dense),
- anisotropic (M depends on ξ not necessarily via $|\xi|$).

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Chlebicka, A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces, Nonl. Analysis 2018.

Anisotropy

Orthotropic functions

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(R1) for $k > 0$, $DT_k(u) \in L_M$ and $\mathcal{A}(x, DT_k(u)) \in L_{M^*}$

(R2) for compactly supported $h \in W^{1,\infty}(\mathbb{R})$ & $\varphi \in V_0^M \cap L^\infty$, we have

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(R3) $\int_{\{l<|u|<l+1\}} \mathcal{A}(x,Du)\cdot Du\,dx \to 0$ as $l\to\infty$.

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Modular density

Orlicz case (Gossez, Studia Math.'82)

C_c^∞ is dense in an Orlicz–Sobolev space with respect to sequential modular closure, not the norm one.
The closures concide (only) in doubling case (i.e. reflexive).

Modular convergence

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Musielak–Orlicz case

To get modular density of C_c^∞ in a Musielak–Orlicz–Sobolev space one needs to impose a balance condition controlling modulus of continuity of function $M(x,\xi)$ and its growth for large ξ.
[Ahmida, C, Gwiazda, Youssfi, JFA'2018], [Gwiazda, Skrzypczak (C), Zatorska-Golstein, JDE'2018], [Borowski, C, submitted'2021]

Examples of spaces with functions that cannot be approximated

Variable exponent spaces

$$W^{1,p(\cdot)} = \{f \in W^{1,1}_{loc} : \ |Df|^{p(x)} \in L^1\}$$

when the exponent p is not log-Hölder continuous

the condition is essentially sharp [Zhikov, Russ.J.Math.Phys.'1995]

Double phase spaces

$$\{f \in W^{1,1}_{loc} : \ |Df|^p + a(x)|Df|^q \in L^1\}$$

with $a : \Omega \to [0, \infty)$, $a \in C^{0,\alpha}$, when powers do not satisfy $q/p \leqslant 1 + \alpha/n$

range is sharp due to [Esposito, Leonetti, & Mingione, JDE'2004]

For more: [Balci, Diening, & Surnachev, CalcVar'2020],

[Colombo & Mingione, ARMA'2015 & more]

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Existence of renormalized solutions

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* for existence of renormalized solutions to a parabolic problem we need also more delicate approximation

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- Under a balance assumption (M) we prove modular density of smooth functions $\Longrightarrow$ existence of $(T_k u)_\delta$.
  Optimality of (M) is of great importance and is not known.
- Monotonicity arguments & Young measures.

* for existence of renormalized solutions to a parabolic problem we need also more delicate approximation

What about other notions of solutions?

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What about other notions of solutions?

Can one prove regularity of very weak solutions?

Methods

Regularity for measure data problems with doubling Orlicz growth

C, Gianneti, Zatorska–Goldstein, preprint'2020, arbitrary $\mu \geqslant 0$

For an equation modelled upon $-\mathrm{div}\left(a(x)g(|Du|)\frac{Du}{|Du|}\right) = \mu$

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$$\mathcal{W}^{\mu}(x_0, R) - R \lesssim u(x_0) \lesssim \inf_{B(x_0,R)} u(x) + \mathcal{W}^{\mu}(x_0, R) + R$$

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C, Youn, Zatorska–Goldstein, preprint'2021, system, arbitrary μ

upper bound only; superquadratic & quasidiagonal operator

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C, NonlAnal'2020, superquadratic operator, special classes of μ

For an equation modelled upon $-\mathrm{div}\left(a(x)g(|Du|)\frac{Du}{|Du|}\right) = \mu$

using harmonic analysis' tools we transfer of regularity from data to a gradient of solution in the relevant scale (Lorentz–Marcinkiewicz–Morrey)

$$\begin{aligned}
\textit{Lorentz\&Marcinkiewicz} \quad \mu \in L(q,s) &\implies g(|Du|) \in L\left(\tfrac{nq}{n-q}, s\right),\ s \in (0,\infty] \\
\textit{Morrey} \quad \mu \in L^{q,\theta} &\implies g(|Du|) \in L^{\frac{\theta q}{\theta - q},\theta} \\
\mu \in L\log L &\implies g^{\frac{n}{n-1}}(|Du|) \in L^1 \\
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C, Israel J. Math.'2019

regularizing effect of a lower order term satisfying sign condition

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Anisotropic methods

Regularity for measure data problems built upon $-\mathrm{div}\left(a(x)\Phi(Du)\frac{1}{|Du|^2}\right) = \mu$

Alberico, C, Cianchi, Zatorska–Goldstein, CalcVar'2019

Anisotropic operator of general Orlicz growth, arbitrary μ

- we detect the sharp range of data for existence of weak solutions
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- using optimal embedding of fully anisotropic Orlicz-Sobolev space into an rearrangement invariant space [Cianchi'2000] we get Marcinkiewicz-type regularity of a very weak solution and its gradient

$$u \in L^{\vartheta(\cdot),\infty}(\Omega) \qquad \text{and} \qquad \Phi(Du) \in L^{\varrho(\cdot),\infty}(\Omega),$$

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We were interested in existence & regularity to

$$(ELL) \qquad -\operatorname{div}\mathcal{A}(x, Du) = \mu \quad \text{in} \quad \Omega \subset \mathbb{R}^N$$

where $\mathcal{A}(x, \xi)$ has growth given by $M(x, \xi)$

$$(PARA) \qquad \partial_t u - \operatorname{div}\mathcal{A}(t, x, Du) = \mu \quad \text{in} \quad \Omega \subset \mathbb{R}^N$$

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C, A pocket guide..., Nonl. Analysis 2018.

Thank you for your attention!

see https://www.mimuw.edu.pl/~ichlebicka/publications