Discontinuous ground states for the NLSE on $\mathbb{R}$ with a Fülöp-Tsutsui δ interaction
(A variational approach)

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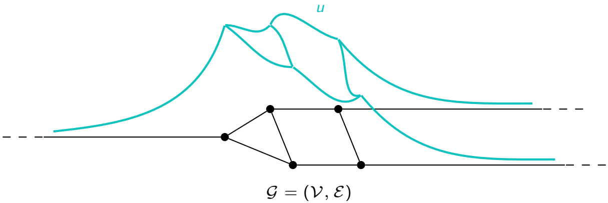
Introduction and setting

- A metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a **connected** structure made of either finite or infinite **edges**, meeting at **vertices**. Each **bounded** edge $e \in \mathcal{E}$ can be identified with an interval $[0, \ell_e]$ and each **unbounded** one with a halfline $[0, +\infty)$.
- $u \in L^p(\mathcal{G})$ is a L^p-function on every edge, while $u \in H^1(\mathcal{G})$ is a H^1-function on every edge and continuous in all vertices.



Kirchhoff \ non-Kirchhoff conditions: an overview

Standing waves of

$$i\partial_t \psi = \mathbf{H}\psi - |\psi|^{2\sigma}\psi$$

with $\sigma > 0$, are solutions of the form $\psi(t,x) = u(x)e^{i\omega t}$ where

$$u'' + |u|^{2\sigma} u = \omega u \quad \text{on every edge,}$$

coupled with the **Kirchhoff** boundary conditions:

$$\begin{cases} u_{e_1}(v) = u_{e_2}(v), & \forall e_1, e_2 \succ v, \forall v \in \mathcal{V} \\ \sum_{e \succ v} \frac{du}{dx_e} = 0, & \forall v \in \mathcal{V}. \end{cases}$$

Non-Kirchhoff boundary conditions:

- δ, δ', Kedem-Katchalsky, dipole, Fülöp-Tsutsui δ;
- nonlinear δ;

Fülöp-Tsutsui δ interaction

$$i\partial_t \psi = H_{\tau,v}\psi - |\psi|^{2\sigma}\psi$$

where $\sigma > 0$ and $\mathbf{H}_{\tau,\mathbf{v}}$ is defined on the domain

$$D(H_{\tau,v}) := \{u \in H^2(\mathbb{R}\backslash\{0\}) : u(0+) = \tau u(0-), \\ u'(0-) - \tau u'(0+) = v u(0-)\}$$

and its action reads $(H_{\tau,v}u)(x) = -u''(x)$ out of the origin, $\tau \in \mathbb{R}\backslash\{0, \pm\mathbf{1}\}$ and $\mathbf{v} > \mathbf{0}$.

The energy space associated is

$$H^1_\tau := \{u \in H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+) \ : \ u(0+) = \tau u(0-)\}$$

and the energy functional is defined as

$$E(u) = \frac{1}{2}\left(||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}\right) - \frac{1}{2\sigma+2}||u||^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R})} - \frac{v}{2}|u(0-)|^2$$

Variational setting of the problem

We are looking for the ground states of the action functional

$$S_\omega(u) = E(u) + \frac{\omega}{2}||u||_2^2,$$

among all functions in H^1_τ on the the Nehari manifold $I_\omega(u) = 0$, where

$$I_\omega(u) = ||u'||_2^2 - ||u||_{2\sigma+2}^{2\sigma+2} - \nu|u(0-)|^2 + \omega||u||_2^2.$$

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Remark $\quad I_\omega(u) = 2S_\omega(u) - \frac{\sigma}{\sigma+1}||u||_{2\sigma+2}^{2\sigma+2}$

$$S_\omega(u) = \widetilde{S}(u) \text{ on the Nehari manifold}$$

with $\widetilde{S}(u) := \frac{\sigma}{2(\sigma+1)}||u||_{2\sigma+2}^{2\sigma+2}$.

About the existence of the ground states

Theorem: for any $\omega > \frac{v^2}{(\tau^2+1)^2}$ there exists $u \in H^1_\tau \setminus \{0\}$ that minimizes $S_\omega(u)$ with the constraint $I_\omega(u) = 0$.

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Proof:

1. thanks to $S_\omega = \widetilde{S}$:

$$\begin{aligned} d(\omega) :&= \inf\{S_\omega(u) \ : \ u \in H^1_\tau \backslash \{0\}, I_\omega(u) = 0\} \\ &= \inf\{\widetilde{S}(u) \ : \ u \in H^1_\tau \backslash \{0\}, I_\omega(u) \leq 0\}, \end{aligned}$$

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4. $d(\omega) < d^0(\omega)$, where $d^0(\omega)$ is the infimum for the problem without punctual interaction (dipole problem, Adami-Noja-Visciglia 2013).

About the stationary states

A stationary state for S_ω constrained on $I_\omega(u) = 0$ solves

$$\begin{cases} -u'' - |u|^{2\sigma} u + \omega u = 0 \quad \text{on every edge} \\ u(0_+) = \tau u(0_-) \\ u'(0_-) - \tau u'(0_+) = v u(0_-) \end{cases} \tag{1}$$

where the conditions at the origin are called **Fülöp-Tsutsui δ type condition**.

Stationary states have the form

$$u_\omega(x) = \begin{cases} \phi_\omega(x + x_-), & x \in \mathbb{R}_- \\ \phi_\omega(x + x_+), & x \in \mathbb{R}_+ \end{cases}$$

where $\phi_\omega(x) := (\omega(\sigma+1))^{\frac{1}{2\sigma}} \cosh^{-\frac{1}{\sigma}}(\sigma\sqrt{\omega}x)$.

Geometric sketch of the stationary states

Thanks to the change of variable $T_\pm = \tanh(\sigma\sqrt{\omega}x_\pm)$, it holds an equivalent form of (1):

$$\begin{cases} T_+ = \frac{1}{\tau^2}\left(T_- + \frac{v}{\sqrt{\omega}}\right) \\ \frac{{T_-}^2}{1-\frac{1}{\tau^{2\sigma}}} - \frac{{T_+}^2}{\tau^{2\sigma}-1} = 1 \end{cases}$$

- for $\omega \leq \frac{v^2}{(\tau^2+1)^2}$, no solutions;
- for $\frac{v^2}{(\tau^2+1)^2} < \omega \leq \frac{v^2}{(\tau^2-1)^2}$, a unique solution (u^L_ω);
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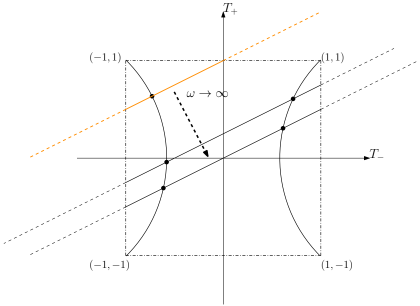

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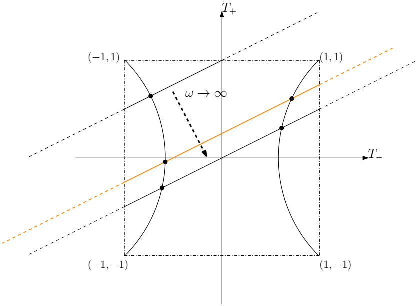

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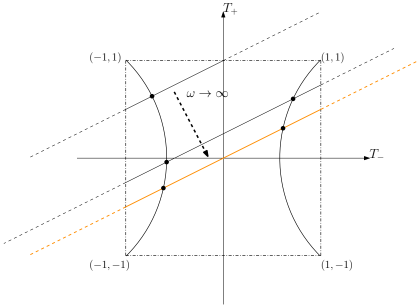
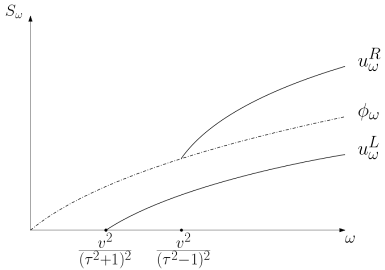
- for $\omega > \frac{v^2}{(\tau^2-1)^2}$, two solutions $(u_\omega^L,\ u_\omega^R)$.



$S_\omega(u) = \widetilde{S}(u)$ on the Nehari manifold

Qualitative graph of bifurcation for the stationary states depending on ω.

NOTE: $\phi_\omega \notin H^1_\tau$!



Theorem: let $\omega > \frac{v^2}{(\tau^2+1)^2}$. The **ground state** is u_ω^L.

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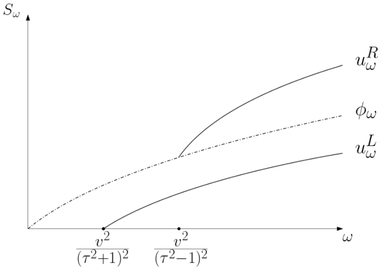
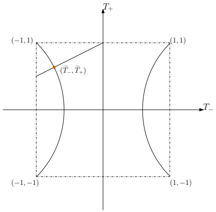


Bumps and tails: identification of the ground state

The stationary state u_ω^L (left branch of the hyperbola)



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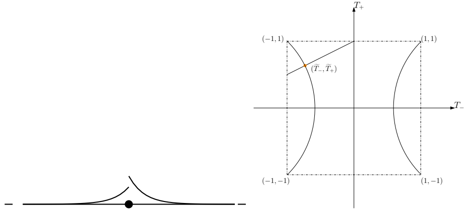

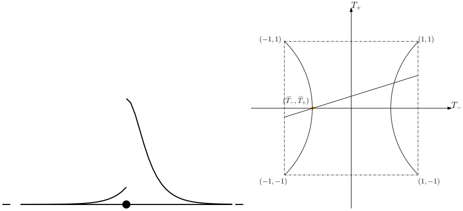

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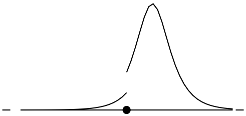

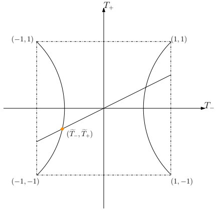

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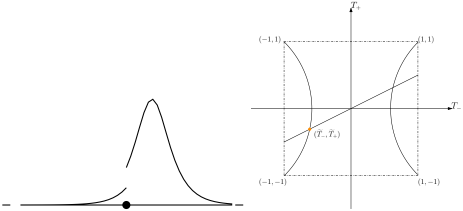
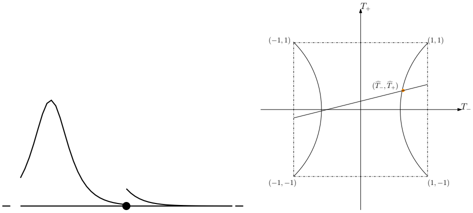


The stationary state u_ω^R (right branch of the hyperbola)

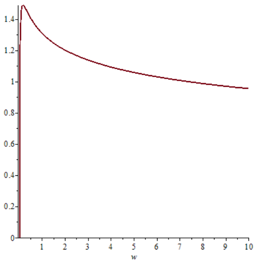


Orbital stability of the ground states

Theorem (stability): let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then for $\sigma \in (\mathbf{0}, \mathbf{2}]$ the ground state u_ω^L is orbitally stable.

We **conjecture** that for $\sigma > 2$, the ground state u_ω^L is stable up to a critical value of ω and then, it becomes unstable.

The mass of u_ω^L depending on ω, for $\sigma = 3$ and $v = 1$, $\tau = 2$.



Thank you!