Strongly Deza graphs

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Deza graphs: generalization of *SRG*

SRG

Let $G = (V, E)$ be a regular graph with n vertices and degree k. Then G is a strongly regular graph with parameters $srg(n, k, \lambda, \mu)$ if:

- every two adjacent vertices have λ common neighbours
- every two non-adjacent vertices have μ common neighbours.

Original paper by Raj Chandra Bose, 1963

R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* 13 (1963) 389–419.

From structural point of view:

A Deza graph is a generalization of a strongly regular graph such that the number of common neighbours of any pair of distinct vertices in a Deza graph does not depend on the adjacency.

Deza graphs: generalization of *SRG*

Definition 1

A Deza graph G with parameters (n, k, b, a) is a k-regular graph of order n for which the number of common neighbours of two vertices takes values b or a, where $b \geqslant a$, whenever G is not the complete or the edgeless graph.

Original paper by Deza & Deza, 1994

A. Deza, M. Deza, The ridge graph of the metric polytope and some relatives. In: *Polytopes: Abstract, Convex and Computational*. NATO ASI Series, Vol. 440 (1994) 359–372, Springer.

Founding Deza graph basics, 1999

M. Erickson, S. Fernando, W.H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, *J. Combinatorial Design*, 7 (1999) 359–405.

Deza graphs vs *SRG*: structural properties

A Deza graph has diameter at least two. A Deza graph of diameter two which is not a strongly regular graph is called a *strictly Deza graph*.

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Strictly Deza graph with parameters $(8, 4, 2, 1)$

Deza graphs: matrix representation

Definition 2

Let G be a graph on n vertices with adjacency matrix M. Then G is a Deza graph with parameters (n, k, b, a) if and only if $M^2 = aA + bB + kI$ for some symmetric $(0,1)$-matrices A and B such that $A + B + I = J$, where J is the all-ones matrix and I is the identity matrix.

$\diamond$ Any Deza graph with parameters (n, k, b, a) is a strongly regular graph with parameters (n, k, λ, μ) if and only if $M = A, M = B$ or $b = a$.

Deza graphs: children

A and B are adjacency matrices of graphs, and the corresponding graphs G_A and G_B are the *children* of G.

Motivation: Yichang, China, August 2019, G2D2

W. Haemers asked on Deza graphs with strongly regular children.

Strongly Deza graphs: a new concept

Definition

A *strongly Deza graph* is a Deza graph G with parameters (n, k, b, a) whose children are strongly regular graphs.

First considering, 2021

V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Generalised dual Seidel switching and Deza graphs with strongly regular children, *Discrete Mathematics*, 344(3) (2021) 112238.
https://doi.org/10.1016/j.disc.2020.112238

Definition and spectral characterization, 2021+

S. Akbari, W. H. Haemers, M. A. Hosseinzadeh, V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Spectra of strongly Deza graphs, 2021.
https://arxiv.org/abs/2101.06877

Deza graphs vs SRG : spectral characterization

From spectral point of view:

- Any strongly regular graph has exactly three distinct eigenvalues.
- A Deza graph can have more than three distinct eigenvalues.

Example

The hypercube graph H_n is a Deza graph with parameters $(2^n, n, 2, 0)$. Its diameter is n. Its spectrum is defined by eigenvalues $n - 2k$ with the multiplicities $\binom{n}{k}$, where $0 \leqslant k \leqslant n$.

The main goal of this talk is

♢ to present a spectral characterization of strongly Deza graphs

Deza graphs and their children: spectrum

Theorem. [Akbari-Ghodrati-Hosseinzadeh-Kabanov-Konstantinova-Shalaginov, Spectra of Deza graphs, *Linear and Multilinear Algebra*, 2020]

Let G be a Deza graph with parameters (n, k, b, a), $b > a$. Let M, A, B be the adjacency matrices of G and its children, respectively. If $\theta_1 = k, \theta_2, \dots, \theta_n$ are the eigenvalues of M, then
(i) the eigenvalues of A are

$$\alpha = \frac{b(n-1) - k(k-1)}{b-a},\ \alpha_2 = \frac{k - b - \theta_2^2}{b-a},\ \dots,\ \alpha_n = \frac{k - b - \theta_n^2}{b-a}.$$

(ii) the eigenvalues of B are

$$\beta = \frac{a(n-1) - k(k-1)}{a-b},\ \beta_2 = \frac{k - a - \theta_2^2}{a-b},\ \dots,\ \beta_n = \frac{k - a - \theta_n^2}{a-b}.$$

Strongly Deza graphs: spectral characterization

By Theorems above, a strongly Deza graph has at most three distinct absolute values of its eigenvalues. But we can be more precise.

Theorem (Spectral characterization-I)

Let G be a strongly Deza graph with parameters (n, k, b, a).

(i) *G has at most five distinct eigenvalues.*

(ii) *If G has two distinct eigenvalues, then $a = 0$, $b = k - 1 \geqslant 1$, and G is a disjoint union of cliques of order $k + 1$.*

(iii) *If G has three distinct eigenvalues, then*
- G is a strongly regular graph with parameters (n, k, λ, μ), where $\{\lambda, \mu\} = \{a, b\}$; or
- G is disconnected and each component is a strongly regular graph with parameters (v, k, b, b); or
- each component is a complete bipartite graph $K_{k,k}$ with $k \geqslant 2$.

Strongly Deza graphs: spectral characterization

Theorem (Spectral characterization-II)

Let G be a connected Deza graph with parameters (n, k, b, a), $b > a$, and it has at most three distinct absolute values of its eigenvalues.

(i) *If G is a non-bipartite graph, then G is a strongly Deza graph.*

(ii) *If G is a bipartite graph, then 1) either G is a strongly Deza graph; 2) or its halved graphs are strongly Deza graphs.*

Definition

If G is a bipartite graph, then the *halved* * graphs of G are two connected components of the graph on the same vertex set, where two vertices are adjacent whenever they are at distance two in G.

* A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989). pp. 25, 438.

Strongly regular graphs: spectrum

Theorem (*SRG*-spectral characterization)

Let G be $srg(n, k, \lambda, \mu)$ and the eigenvalues k, r, and s. Then:

(i) *The principal eigenvalue k has the multiplicity 1.*

(ii) *The restricted integer eigenvalues*

$$r, s = \frac{(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

have the multiplicities $f, g = \frac{1}{2}\left(n - 1 \mp \frac{(r + s)(n - 1) + 2k}{r - s}\right)$.

(iii) *If r and s are not integers, then*

$$r, s = \frac{-1 \pm \sqrt{n}}{2}$$

with the same multiplicities.

Integral strongly Deza graphs

The next theorem gives some conditions on an integral strongly Deza graph with respect to eigenvalues of its children.

Theorem

Let G be a strongly Deza graph with parameters (n,k,b,a). Let its child G_A be a strongly regular graph with parameters (n,α,λ,μ) and eigenvalues α, r, s with multiplicities $1, f, g$. If M is the adjacency matrix of G with spectrum $\{k^1, \theta_2^{m_2}, \theta_3^{m_3}, \theta_4^{m_4}, \theta_5^{m_5}\}$, then one of the statements hold:

(i) $\theta_2^2 = k-b-s(b-a)$ *and* $\theta_3^2 = k-b-r(b-a)$ *are squares; in this case G is an integral graph.*

(ii) $\theta_2^2 = k-b-s(b-a)$ *is not a square; then* $\theta_3^2 = k-b-r(b-a)$ *is a nonzero square and* $m_2 = m_5 = f/2$.

(iii) $\theta_3^2 = k-b-r(b-a)$ *is not a square; then* $\theta_2^2 = k-b-s(b-a)$ *is a nonzero square and* $m_3 = m_4 = g/2$.

This theorem is a generalization of Theorem 2.2 in [HKhM,DDG,2011].

Integral strongly Deza graphs

Moreover,

Corollary

The children of a strongly Deza graph are integral graphs.

A graph is said to be *singular* if and only if zero is its eigenvalue.

Theorem

Any singular strongly Deza graph is an integral graph with four distinct eigenvalues.

There are infinitely many singular strongly Deza graphs arising from the affine group $\mathbb{A}\mathrm{ff}(1, \mathbb{F}_{q^t})$, for any prime power q and $t > 1$, as divisible design Cayley graphs (see Kabanov, Shalaginov, On divisible design Cayley graphs, ADAM, `https://doi.org/10.26493/2590-9770.1340.364`).
The smallest example is known as the line graph of the octahedron. It has parameters $(12, 6, 3, 2)$ and spectrum $\{6^1, 2^3, 0^2, (-2)^6\}$.

Thank you for attention!

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