Minimum supports of eigenfunctions of graphs

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Basic definitions

Let $G = (V, E)$ be a graph and let λ be an eigenvalue of G. The set of neighbors of a vertex x is denoted by $N(x)$. A function $f : V \longrightarrow \mathbb{R}$ is called a λ-eigenfunction of G if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any vertex $x \in V$. The support of a function $f : V \longrightarrow \mathbb{R}$ is the set $S(f) = \{x \in V \mid f(x) \neq 0\}$. A λ-eigenfunction of G is called optimal if it has the minimum cardinality of the support among all λ-eigenfunctions of G.

Main problem

MS-problem

Let G be a graph and let λ be an eigenvalue of G. Find the minimum cardinality of the support of a λ-eigenfunction of G.

MS-problem is closely related to the intersection problem of two combinatorial objects and to the problem of finding the minimum cardinality of combinatorial trades [1,2].

[1] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, To the theory of q-ary Steiner and other-type trades, Discrete Mathematics 339(3) (2016) 1150–1157.

[2] E. Sotnikova, A. Valyuzhenich, Minimum supports of eigenfunctions of graphs: a survey, February 2021, arXiv:2102.11142v1.

Results on MS-problem

MS-problem has been studied for the following families of graphs:

- bilinear forms graphs (Sotnikova, 2019)
- cubical distance-regular graphs (Sotnikova, 2018)
- Doob graphs (Bespalov, 2018)
- Grassmann graphs (Cho 1999; Krotov, Mogilnykh, Potapov, 2016)
- Hamming graphs (Vorob'ev, Krotov, 2014; Krotov 2016; Valyuzhenich, Vorobev, 2019; Valyuzhenich 2021)
- Johnson graphs (Vorob'ev, Mogilnykh, Valyuzhenich, 2018)
- Paley graphs (Goryainov, Kabanov, Shalaginov, Valyuzhenich, 2018)
- Star graphs (Goryainov, Kabanov, Konstantinova, Shalaginov, Valyuzhenich, 2020)

Hamming graph

Let $\Sigma_q = \{0, 1, \ldots, q-1\}$. The Hamming graph $H(n,q)$ is defined as follows:

- the vertex set of $H(n,q)$ is Σ_q^n
- two vertices are adjacent if they differ in exactly one coordinate

The Hamming graph $H(n,q)$ has $n+1$ distinct eigenvalues $\lambda_i(n,q) = n(q-1) - q \cdot i$, where $0 \leq i \leq n$.

MS-problem for the Hamming graph

- MS-problem for the Hamming graph $H(n, 2)$ is solved for all eigenvalues in [3].
- MS-problem for the Hamming graph $H(n, q)$ is solved for all eigenvalues and $q \geq 3$ in [4,5].
- Moreover, in [4] a characterization of optimal $\lambda_i(n, q)$-eigenfunctions of $H(n, q)$ was obtained for $q \geq 3$, $i \leq \frac{n}{2}$ and $q \geq 5$, $i > \frac{n}{2}$.

[3] D. S. Krotov, Trades in the combinatorial configurations, XII International Seminar Discrete Mathematics and its Applications, Moscow, 20–25 June 2016, 84–96.

[4] A. Valyuzhenich, K. Vorob'ev, Minimum supports of functions on the Hamming graphs with spectral constraints, Discrete Mathematics 342(5) (2019) 1351–1360.

[5] A. Valyuzhenich, Eigenfunctions and minimum 1-perfect bitrades in the Hamming graph, Discrete Mathematics 344(3) (2021) 112228.

Optimal eigenfunctions of the Hamming graph

So, the problem of a characterization of optimal $\lambda_i(n, q)$-eigenfunctions of $H(n, q)$ is open for the following cases:

- $q = 2$
- $q = 3$ and $i > \frac{n}{2}$
- $q = 4$ and $i > \frac{n}{2}$

MS-problem for the Hamming graph $H(n, 2)$

In 2016 D. Krotov [3] proved that the minimum cardinality of the support of a $\lambda_i(n, 2)$-eigenfunction of $H(n, 2)$ is $\max(2^i, 2^{n-i})$. He also gave examples of optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$. These optimal eigenfunctions can be constructed as a tensor product of several optimal eigenfunctions defined on the vertices of the Hamming graphs of diameter not greater than two.

[3] D. S. Krotov, Trades in the combinatorial configurations, XII International Seminar Discrete Mathematics and its Applications, Moscow, 20–25 June 2016, 84–96.

The tensor product of functions

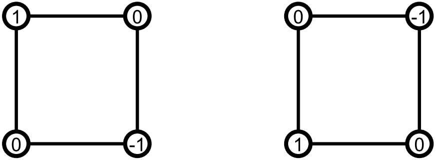
Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs. Let $f_1 : V_1 \longrightarrow \mathbb{R}$ and $f_2 : V_2 \longrightarrow \mathbb{R}$. Denote $G = G_1 \Box G_2$. We define the tensor product $f_1 \cdot f_2$ on the vertices of G by the following rule:

$$(f_1 \cdot f_2)(x, y) = f_1(x) f_2(y)$$

for $(x, y) \in V(G) = V_1 \times V_2$.

Optimal eigenfunctions in $H(2,2)$ and $H(1,2)$

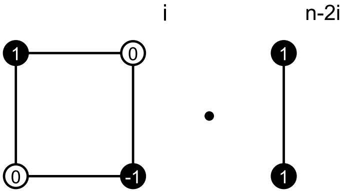
The set A consists of two optimal 0-eigenfunctions of $H(2,2)$:



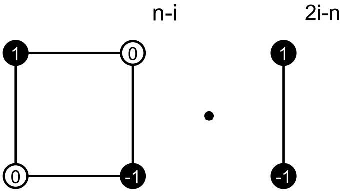
The sets B and C:

Optimal eigenfunctions in $H(n, 2)$

Optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$ for $i \leq \frac{n}{2}$:



Optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$ for $i > \frac{n}{2}$:



Main result

Theorem (V., 2021)

- If $i \leq \frac{n}{2}$, then any optimal $\lambda_i(n,2)$-eigenfunction of $H(n,2)$ is the tensor product of i functions from the set A and $n-2i$ functions from the set B up to a permutation of coordinate positions and the multiplication by a scalar.
- If $i > \frac{n}{2}$, then any optimal $\lambda_i(n,2)$-eigenfunction of $H(n,2)$ is the tensor product of $n-i$ functions from the set A and $2i-n$ functions from the set C up to a permutation of coordinate positions and the multiplication by a scalar.

Thank you for your attention!