Novák's conjecture on cyclic Steiner triple systems and its generalization

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Cyclic 2-designs

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block orbit 1 $\{0,1,4\}$, $\{1,2,5\}$, $\{2,3,6\}$, $\{3,4,7\}$, $\{4,5,8\}$, $\{5,6,9\}$, $\{6,7,10\}$, $\{7,8,11\}$, $\{8,9,12\}$, $\{0,9,10\}$, $\{1,10,11\}$, $\{2,11,12\}$, $\{0,3,12\}$;

block orbit 2 $\{0,2,7\}$, $\{1,3,8\}$, $\{2,4,9\}$, $\{3,5,10\}$, $\{4,6,11\}$, $\{5,7,12\}$, $\{0,6,8\}$, $\{1,7,9\}$, $\{2,8,10\}$, $\{3,9,11\}$, $\{4,10,12\}$, $\{0,5,11\}$, $\{1,6,12\}$.

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- The blocks of a cyclic (v, k, λ)-design can be partitioned into orbits under $\mathbb{Z}_v$. Choose any fixed block from each orbit and then call them base blocks.

Novák's conjecture on cyclic Steiner triple systems

Conjecture (Novák, 1974)

For any cyclic STS(v) with $v \equiv 1 \pmod 6$, it is always possible to choose one block from each block orbit so that the chosen blocks are pairwise disjoint.

[a] R.J.R. Abel, M. Buratti, Difference families, in: C.J. Colbourn, J.H. Dinitz, Handbook of Combinatorial Designs (2nd Edition), CRC Press, 2006, 392–410.

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- Novák's Conjecture is widely believed to be true but not much progress has been made on it (see also Remark 16.22 in [a] or Work point 22.5.2 in [b]).

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- It is known that Novák's Conjecture holds for all $v \equiv 1 \pmod 6$ and $v \leqslant 61$.

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Cyclic difference families

- A (v, k, λ)-cyclic difference family (CDF) is a family $\mathcal{F}$ of k-subsets (called base blocks) of $\mathbb{Z}_v$ such that the multiset

$$\Delta\mathcal{F} := \{x - y : x, y \in F, x \neq y, F \in \mathcal{F}\}$$

contains every element of $\mathbb{Z}_v \setminus \{0\}$ exactly λ times. It consists of $\lambda(v-1)/(k(k-1))$ base blocks.

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- A (v, k, λ)-CDF $\mathcal{F} \Rightarrow$ a cyclic (v, k, λ)-design with block-multiset

$$dev\mathcal{F} := \{F + t : F \in \mathcal{F}, t \in \mathbb{Z}_v\}.$$

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  The converse is usually not true. But when $\gcd(v, k) = 1$, $\mathcal{F}$ is a (v, k, λ)-CDF $\Leftrightarrow$ $dev\mathcal{F}$ is a cyclic (v, k, λ)-design.

Disjoint difference families

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Remark

Dinitz and Rodney [a] proved that a $(v, 3, 1)$-DDF exists for any $v \equiv 1 \pmod 6$ by taking a suitable $(v, 3, 1)$-CDF and then replacing each of its base blocks B_i by a suitable translate $B_i + t_i$.

[a] J.H. Dinitz, P. Rodency, Disjoint difference families with block size 3, Util. Math., 52 (1997), 153–160.

Karasev and Petrov's Theorem

Theorem

Let $\mathbb{F}$ be an arbitrary field, and let m and d be positive integers such that $(md)!/(d!)^m \neq 0$ in $\mathbb{F}$. Let $X_1, \ldots, X_m$ and $T_1, \ldots, T_m$ be subsets of $\mathbb{F}$ such that

$$(1)\ \forall i < j \quad |X_i - X_j| \leqslant 2d, \qquad (2)\ \forall i \quad |T_i| \geqslant (m-1)d + 1,$$

where $X_i - X_j := \{x - y : x \in X_i, y \in X_j\}$. Then there exists a system of representatives $t_i \in T_i$ such that the sets $X_1 + t_1, \ldots, X_m + t_m$ are pairwise disjoint [a].

[a] R.N. Karasev, F.V. Petrov, Partitions of nonzero elements of a finite field into pairs, Israel Journal of Mathematics, 192 (2012), 143–156.

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- Apply the above theorem with $m = (p-1)/6$ and $d = 5$, where $p \equiv 1 \pmod 6$ is a prime.

Application of Karasev and Petrov's Theorem

Theorem

Let $k \geqslant 2$ and p be a prime. Every cyclic $(p, k, 1)$-design is generated by a $(p, k, 1)$-DDF [a].

[a] T. Feng, D. Horsley, and X. Wang, Novák's conjecture on cyclic Steiner triple systems and its generalization, arXiv:2001.06995.

Known results on cyclic $(p, k, 1)$-design with p a prime

Let $p \equiv 1 \pmod{k(k-1)}$ be a prime.

1. There exists a $(p, k, 1)$-CDF for $k \in \{4, 5$[c] [d], 6[e]$\}$ and $(k, p) \neq (6, 61)$.
2. There exists a $(p, k, 1)$-CDF whenever $p > \binom{k}{2}^{k(k-1)}$ [f].

[c] M. Buratti, Constructions for $(q, k, 1)$ difference families with q a prime power and $k = 4, 5$, Discrete Math. 138 (1995), 169–175.

[d] K. Chen and L. Zhu, Existence of $(q, k, 1)$ difference families with q a prime power and $k = 4, 5$, J. Combin. Des., 7 (1999), 21–30.

[e] K. Chen and L. Zhu, Existence of $(q, 6, 1)$ difference families with q a prime power, Des. Codes Crypt., 15 (1998), 167–174.

[f] R.M. Wilson, Cyclotomy and difference families in elementary abelian groups, J. Number Theory, 4 (1972), 17–47.

A generalization of Novák's conjecture

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For any cyclic $(v, k, 1)$-design, it is always possible to choose one block from each block orbit so that the chosen blocks are pairwise disjoint.

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- The above conjecture, if true, would reduce the existence of $(v, k, 1)$-DDFs to the existence of $(v, k, 1)$-CDFs.

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Applications of DDFs

Frequency hopping sequences, self-synchronising codes, splitting A-codes, secret sharing schemes with cheater detection, algebraic manipulation detection codes, and high-rate quasi-cyclic codes [a].

[a] S. Ng, M.B. Paterson, Disjoint difference families and their applications, Des. Codes Cryptogr., 78 (2016), 103–127.

Sketch of the proof on Karasev and Petrov's Theorem

Theorem (Karasev and Petrov)

Let $(md)!/(d!)^m \neq 0$ in $\mathbb{F}$ and

$$\forall i < j \quad |X_i - X_j| \leqslant 2d, \qquad \forall i \quad |T_i| \geqslant (m-1)d + 1.$$

Then there exists a system of representatives $t_i \in T_i$ such that the sets $X_1 + t_1, \ldots, X_m + t_m$ are pairwise disjoint.

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- Set $X_{ij} = X_i - X_j$.
- Write $f(t_1, \ldots, t_m) = \prod_{1 \leqslant i < j \leqslant m} \prod_{x \in X_{ij}} (t_i - t_j - x)$.

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- Write $f(t_1, \ldots, t_m) = \prod_{1 \leqslant i < j \leqslant m} \prod_{x \in X_{ij}} (t_i - t_j - x)$.
- If f attains a nonzero value on $T_1 \times \cdots \times T_m$ then the proof is complete.

Combinatorial Nullstellensatz

Theorem

Assume that

1. a polynomial $f(x_1, x_2, \ldots, x_n)$ over a field $\mathbb{F}$ has degree at most $c_1 + c_2 + \cdots + c_n$, where c_i are non-negative integers, and denote by C the coefficient at $x_1^{c_1} \cdots x_n^{c_n}$ in f (maybe, $C = 0$);
2. $A_1, A_2, \ldots, A_n$ be arbitrary subsets of $\mathbb{F}$ such that $|A_i| = c_i + 1$ for any i.

If $C \neq 0$, then there exists a system of representatives $\alpha_i \in A_i$ such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq 0$.

Further generalization of Novák's conjecture

Conjecture 2

Let $k \geqslant \lambda + 1$. There exists an integer v_0 such that, for any cyclic (v, k, λ)-design with $v \geqslant v_0$, it is always possible to choose one block from each block orbit so that the chosen blocks are pairwise disjoint.

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- Compared with Conjecture 1, Conjecture 2 is stated for sufficiently large v.

Remark

A (v, k, λ)-DDF necessarily has $1 \leqslant \lambda \leqslant k - 1$ apart from the trivial case of a (k, k, k)-DDF [a].

[a] M. Buratti, On disjoint $(v, k, k-1)$ difference families, Des. Codes Cryptogr., 87 (2019), 745–755.

Asymptotic solution

A partial parallel class of a (v, k, λ)-design is a set of pairwise disjoint blocks.

Theorem

Let $k \geqslant 2\lambda + 1$ and let $s = \lfloor \frac{k-1}{\lambda} \rfloor$. For each real number $\epsilon > 0$, there is an integer v_0 such that, for each integer $v \geqslant v_0$, any cyclic (v, k, λ)-design with t orbits has a partial parallel class that contains $s - 1$ blocks from each of at most ϵt orbits and contains s blocks from each other orbit.

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- Parameter s:

$$\frac{(v-1)/k}{\lambda(v-1)/k(k-1)} = \frac{k-1}{\lambda}.$$

Sketch of the proof - Preliminaries

- Let $(V, \mathcal{B})$ be a cyclic (v, k, λ)-design with orbits $\mathcal{B}_1, \ldots, \mathcal{B}_t$ and suppose that m of these orbits are full.

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- Let $\mathcal{P}$ be a partial parallel class of $(V, \mathcal{B})$. For any nonnegative integer a, define

$$T_a(\mathcal{P}) = \{i \in [t] : |\mathcal{P} \cap \mathcal{B}_i| = a\}$$

to be the set of indices of orbits of $(V, \mathcal{B})$ that contain exactly a blocks in $\mathcal{P}$, and define

$$\tau_a(\mathcal{P}) = |T_a(\mathcal{P})|.$$

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- Goal: find a partial parallel class $\mathcal{P}''$ of $(V, \mathcal{B})$ such that $\tau_a(\mathcal{P}'') = 0$ for $0 \leqslant a \leqslant s-2$, $\tau_{s-1}(\mathcal{P}'') < \epsilon t$ and $\tau_s(\mathcal{P}'') = t - \tau_{s-1}(\mathcal{P}'')$.

Sketch of the proof - Two steps

- STEP 1: We obtain a partial parallel class $\mathcal{P}$ of $(V, \mathcal{B})$ such that

$$\tau_0(\mathcal{P}) \leqslant \epsilon^* t \text{ and } \tau_s(\mathcal{P}) = t - \tau_0(\mathcal{P}).$$

So $\mathcal{P}$ contains s blocks from almost every block orbit.

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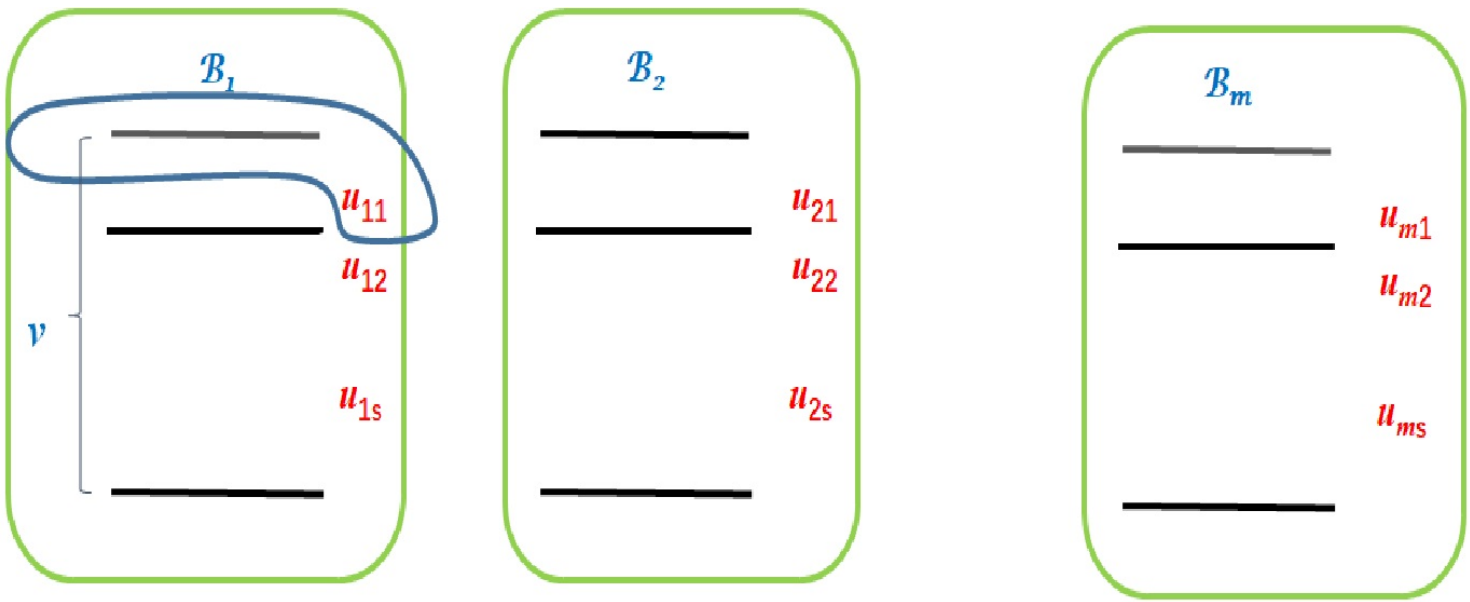
  So $\mathcal{P}$ contains s blocks from almost every block orbit.

- STEP 2: We then prove that if each orbit of $(V, \mathcal{B})$ contains sufficiently many "good blocks" relative to some partial parallel class, then this class can be modified so that it contains s blocks from almost every orbit and $s-1$ blocks from each remaining orbit.

Sketch of the proof - Step 1

- Let $W = \{u_{i,j} : i \in [m], j \in [s]\}$ be a set of vertices disjoint from V.
- Form a $(k+1)$-uniform hypergraph G with vertex set $V \cup W$ and edge set

$$\{B \cup \{u_{i,j}\} : B \in \mathcal{B}_i, i \in [m], j \in [s]\}.$$



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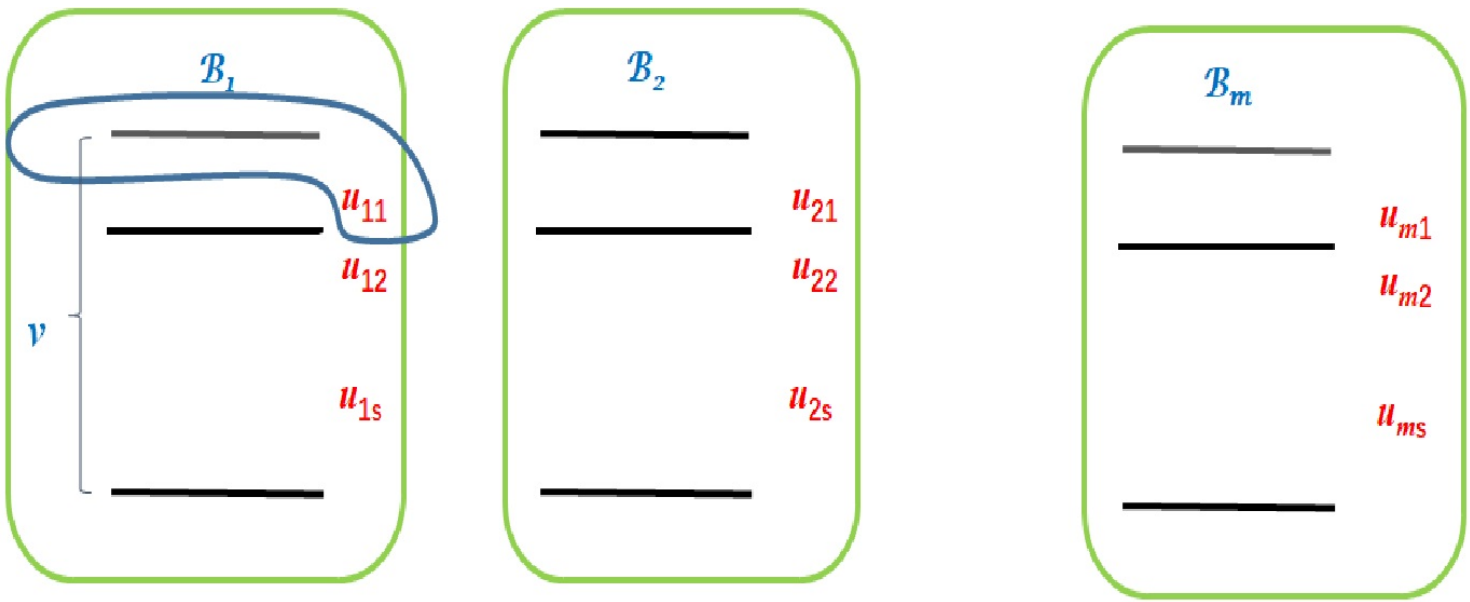

- $\delta_G \geqslant v - k$, $\Delta_G \leqslant v$, and $\Delta_G^c \leqslant k + \lambda - 1$.

Sketch of the proof - Step 1

- By Pippenger and Spencer's theorem on edge-colouring of r-uniform hypergraphs, we shows that G has a proper edge-colouring with $(1 + o(1))v$ colours.
  - (Pippenger and Spencer's Theorem) Every almost regular r-uniform hypergraph G with small maximum codegree can be edge-coloured with close to Δ_G colours.

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- Let $\mathcal{C}$ be a largest colour class of this colouring and let

$$M = \{i \in [m] : |\{j \in [s] : u_{i,j} \text{ is in an edge in } \mathcal{C}\}| = s\}.$$

Then $|M| > (1-\epsilon^*)m$.

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  Then $|M| > (1-\epsilon^*)m$.
- $\mathcal{C}|_M$ gives rise to the required partial parallel class $\mathcal{P}$.

Sketch of the proof - Step 2

- We say that a block $B \in \mathcal{B}$ is $\mathcal{P}$-good if,
  1. for each $i \in T_0(\mathcal{P}) \cup \cdots \cup T_{s-1}(\mathcal{P})$, B intersects no block in $\mathcal{P} \cap \mathcal{B}_i$;
  2. for each $i \in T_s(\mathcal{P})$, B intersects at most one block in $\mathcal{P} \cap \mathcal{B}_i$.

- Careful counting shows that if each orbit of $(V, \mathcal{B})$ contains sufficiently many good blocks relative to some partial parallel class, then this class can be modified so that it contains s blocks from almost every orbit and $s-1$ blocks from each remaining orbit.

Strong Novák's conjecture on cyclic STSs

- A $(v, 3, 1)$-DDF for $v \equiv 1 \pmod 6$ is called symmetric if its base blocks can be chosen in such a way that for any nonzero x of $\mathbb{Z}_v$, at most one of x and its complement $v - x$ occurs in the base blocks and no base block contains zero.

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Every cyclic STS(v) with $v \equiv 1 \pmod 6$ is generated by a symmetric $(v, 3, 1)$-DDF.

- For example: a cyclic STS(13) that implies a $(v, 3, 2)$-DDF:
  - $\{0, 1, 4\}$, $\{1, 2, 5\}$, $\{2, 3, 6\}$, $\{3, 4, 7\}$, $\{4, 5, 8\}$, $\{5, 6, 9\}$, $\{6, 7, 10\}$, $\{7, 8, 11\}$, $\{8, 9, 12\}$, $\{0, 9, 10\}$, $\{1, 10, 11\}$, $\{2, 11, 12\}$, $\{0, 3, 12\}$;
  - $\{0, 2, 7\}$, $\{1, 3, 8\}$, $\{2, 4, 9\}$, $\{3, 5, 10\}$, $\{4, 6, 11\}$, $\{5, 7, 12\}$, $\{0, 6, 8\}$, $\{1, 7, 9\}$, $\{2, 8, 10\}$, $\{3, 9, 11\}$, $\{4, 10, 12\}$, $\{0, 5, 11\}$, $\{1, 6, 12\}$.

Extension of STSs to designs with size four

Theorem

Let $v \equiv 1 \pmod 6$. If there exists a symmetric $(\mathbb{Z}_v, 3, 1)$-DDF, then there exists a $(2v, 2, 4, 1)$-CDF.

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- Let $\{a_i, b_i, c_i\}$, $1 \leqslant i \leqslant (v-1)/6$, be a symmetric $(\mathbb{Z}_v, 3, 1)$-DDF.
- Then

$$\mathcal{F} = \{\{(0,0),(1,a_i),(1,b_i),(1,c_i)\} : 1 \leqslant i \leqslant (v-1)/6\}$$

forms a $(2v, 2, 4, 1)$-CDF over $\mathbb{Z}_2 \times \mathbb{Z}_v \cong Z_{2v}$.

Conclusion

Conjecture 1

For any cyclic $(v, k, 1)$-design, it is always possible to choose one block from each block orbit so that the chosen blocks are pairwise disjoint.

Conjecture 2

Let $k \geqslant \lambda + 1$. There exists an integer v_0 such that, for any cyclic (v, k, λ)-design with $v \geqslant v_0$, it is always possible to choose one block from each block orbit so that the chosen blocks are pairwise disjoint.

Conjecture (Novák, 1974)

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Thanks for your attention!