Characterization of manifolds of constant curvature by spherical curves and ruled surfaces

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Motivation

Guiding question:

To what extent do properties of geometric objects depend on *where* they are defined?

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Curves in a manifold M are uniquely determined up to rigid motions by their curvatures and torsion if and only if M is a space form.

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Curves in a manifold M are uniquely determined up to rigid motions by their curvatures and torsion if and only if M is a space form.

Theorem B [Kulkarni, P. Am. Math. Soc. 53 (1975) 440]

Every sufficiently small geodesic sphere in a manifold M is totally umbilic if and only if M is a space form.

Spherical curves in Euclidean space

Theorem C

A curve is spherical if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\right)\right]+\frac{\tau}{\kappa}=0.$$

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Rotation minimizing (RM) frames [Bishop, Am Math Mon 82 (1975) 246]

A unit vector field v is RM along α' if $\mathrm{v}'=\mu\alpha'$ for some μ. If instead of Frenet we equip α with an RM frame frame $\{\mathrm{t}=\alpha',\mathrm{n}_1,\dots,\mathrm{n}_m\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\mathrm{t}\\ \mathrm{n}_1\\ \vdots\\ \mathrm{n}_m\end{pmatrix}=\begin{pmatrix}0 & \kappa_1 & \cdots & \kappa_m\\ -\kappa_1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\kappa_m & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}\mathrm{t}\\ \mathrm{n}_1\\ \vdots\\ \mathrm{n}_m\end{pmatrix}.$$

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Theorem D [Bishop, Am. Math. Mon. 82 (1975) 246]

Let $\{\mathsf{t}=\alpha',\mathsf{n}_1,\ldots,\mathsf{n}_m\}$ be an RM frame, then α is spherical if and only if

$$a_1\kappa_1(s)+\cdots+a_m\kappa_m(s)=\frac{1}{r}$$

where a_j is constant, $a_1^2+\cdots+a_m^2=1$, and r is the radius of the sphere.

Note: The coefficients a_i are the coordinates of the unit normal of the sphere with respect to $\{\mathsf{n}_1,\ldots,\mathsf{n}_m\}$.

Spherical curves in space forms

Theorem 1 [— & da Silva, Mediterr. J. Math. 15 (2018) 70]

Let α be a C^2 curve in $\mathbb{S}^{m+1}(r)$ or $\mathbb{H}^{m+1}(r)$. Then, α is spherical iff

$$\begin{cases} a_1\,\kappa_1+\cdots+a_m\,\kappa_m+\dfrac{1}{r}\,\cot\left(\dfrac{z_0}{r}\right) & =0\,, \quad \text{if } \alpha\subseteq\mathbb{S}^{m+1}(r) \\ a_1\,\kappa_1+\cdots+a_m\,\kappa_m+\dfrac{1}{r}\,\coth\left(\dfrac{z_0}{r}\right) & =0\,, \quad \text{if } \alpha\subseteq\mathbb{H}^{m+1}(r) \end{cases},$$

for some z_0 (radius) and a_i (coordinates of the unit normal). Moreover, the coefficients $\frac{1}{r}\cot(\frac{z_0}{r})$ and $\frac{1}{r}\coth(\frac{z_0}{r})$ are the mean curvature of the sphere.

Theorem 2 [— & da Silva, Mediterr. J. Math. 15 (2018) 70]

Let α be a C^4 curve in $\mathbb{S}^3(r)$ or $\mathbb{H}^3(r)$. Then, α is spherical iff

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\right)\right]+\frac{\tau}{\kappa}=0\,.$$

Proof of Theorem 1

Let α be spherical. We may write

$$\alpha(s) = \exp_p(z_0\,\mathrm{V}(c_0\,s)) = \cos(\frac{z_0}{r})p + r\sin(\frac{z_0}{r})\mathrm{V}(c_0 s) \Rightarrow \mathrm{t}_\alpha(s) = \mathrm{V}'(c_0 s),$$

where $c_0 = \frac{1}{r}\csc(\frac{z_0}{r})$ and $\mathrm{V} : I \to \mathbb{S}^m(1) \subseteq T_p\mathbb{S}^{m+1}(r)$ has unit speed.

The unit normal ξ along α is the same as the tangent of the radial geodesic $\beta_s(u) = \exp_p(u\mathrm{V}(c_0 s))$ at $u = z_0$. We can show that

$$\nabla_{\mathrm{t}_\alpha}\xi = \frac{1}{r}\cot\left(\frac{z_0}{r}\right)\,\mathrm{t}_\alpha\,.$$

Note: The unit normal ξ is RM along any curve!
Let us write ξ along α as

$$\xi(s) = \mathrm{t}_{\beta_s}(z_0) = a_1(s)\mathrm{n}_1(s) + \cdots + a_m(s)\mathrm{n}_m(s)\,.$$

The coefficient $a_i = \langle \xi, \mathrm{n}_i \rangle$ is constant:

$$a_i' = \langle \nabla_{\mathrm{t}_\alpha} \xi, \mathrm{n}_i \rangle + \langle \xi, \nabla_{\mathrm{t}_\alpha} \mathrm{n}_i \rangle = \langle \frac{1}{r} \cot(\frac{z_0}{r}) \, \mathrm{t}_\alpha, \mathrm{n}_i \rangle + \langle \xi, -\kappa_i \mathrm{t}_\alpha \rangle = 0.$$

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Finally, taking the derivative of $\langle \xi, \mathrm{t}_\alpha \rangle = 0$ along α

$$0 = \langle \nabla_{\mathrm{t}_\alpha} \xi, \mathrm{t}_\alpha \rangle + \langle \xi, \nabla_{\mathrm{t}_\alpha} \mathrm{t}_\alpha \rangle = \langle \frac{1}{r} \cot(\frac{z_0}{r})\, \mathrm{t}_\alpha, \mathrm{t}_\alpha \rangle + \langle \sum_i a_i \mathrm{n}_i, \sum_i \kappa_j \mathrm{n}_j \rangle$$

$$\Rightarrow 0 = \frac{1}{r} \cot(\frac{z_0}{r}) + a_1 \kappa_1(s) + \cdots + a_m \kappa_m(s).$$

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$$\Rightarrow 0 = \frac{1}{r} \cot(\frac{z_0}{r}) + a_1 \kappa_1(s) + \cdots + a_m \kappa_m(s).$$

Conversely, given $\alpha(s)$, define $\mathrm{W}(s) = -a_1 \mathrm{n}_1(s) - \cdots - a_m \mathrm{n}_m(s)$ and

$$p(s) = \cos(\frac{z_0}{r})\alpha(s) - r \sin(\frac{z_0}{r})\mathrm{W}(s).$$

We have $p'(s) = 0$. Then, $p(s)$ is constant and the geodesics starting at α with velocity W travel the same distance to reach P. Therefore, α is a spherical curve. □

Curves on totally umbilical hypersurfaces

Theorem 3 [— & da Silva, Annali di Matematica 199 (2020) 217]

$\Sigma^m \subset M^{m+1}$ is umbilical if and only if its unit normal ξ is RM along every $\alpha : I \to \Sigma$. In addition, every curve in Σ satisfies

$$a_1\kappa_1(s) + \cdots + a_m\kappa_m(s) = H(\alpha(s)),$$

where H is the mean curvature, $\xi = \sum_i a_i \mathrm{n}_i$ with a_i constant and n_i RM.

Thus, $\sum a_i\kappa_i = H$ for spherical curves if and only if M is a space form.

Finally, if we use the Frenet frame, then $\Sigma^2 \subset M^3$ is umbilical if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{H}{\kappa}\right)\right] + \frac{\tau}{\kappa}H = 0\,.$$

Note: The same is valid for pseudo-Riemannian manifolds.

Curves on totally umbilical hypersurfaces

Theorem E [Baek et al, Am. Math. Mon. 110 (2003) 830]

Let $r > 0$ be constant. Then every curve $\alpha : I \to \Sigma^m \subset \mathbb{E}^{m+1}$ satisfies $\kappa \geq \frac{1}{r}$ if and only if Σ^m is a sphere of radius r.

Theorem 4 [— & da Silva, Annali di Matematica 199 (2020) 217]

Every curve $\alpha : I \to \Sigma^m \subset M^{m+1}$ satisfies $\kappa \geq |H|$, where H is the mean curvature of M, if and only if Σ^m is totally umbilical.
Consequently, $\kappa \geq |H|$ characterizes spherical curves iff M is a space form.

Note: The same is valid for pseudo-Riemannian manifolds.

Curves on totally umbilical hypersurfaces

Theorem F [Pansonato & Costa, Geom. Dedicata 136 (2008) 111]

Every simple closed curve $\alpha : I \to \Sigma^2 \subset \mathbb{E}^3$, $\mathbb{S}^3(r)$, or $\mathbb{H}^3(r)$ satisfies $\oint \tau = 0$ if and only if Σ^2 is totally umbilical.

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Theorem 5 [— & da Silva, Annali di Matematica 199 (2020) 217]

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Proof of Thr. B (umbilical spheres $\Leftrightarrow$ space form)

Given $X_q, Y_q \in T_qM$, $\exists\, p$ where $S(p,R)$ is tang. to X_q and normal to Y_q.

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$$q = f(R,0),\ \frac{\partial f}{\partial s}(R,0) = X_q,\ \text{and}\ \frac{\partial f}{\partial u}(R,0) = Y_q.$$

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$$q = f(R, 0),\ \frac{\partial f}{\partial s}(R, 0) = X_q,\ \text{and}\ \frac{\partial f}{\partial u}(R, 0) = Y_q.$$

Extend X_q, Y_q to $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial u}$. Then $\nabla_Y Y = 0$, $[X, Y] = 0$, and $\nabla_X Y = \nabla_Y X$. From the umbilicity, $\nabla_X Y = -\lambda X$, it follows

$$R(Y, X)Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y + \nabla_{[Y,X]} Y = \nabla_Y(\lambda X) = (\frac{\partial \lambda}{\partial u} - \lambda^2)X.$$

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The sectional curvature of span$\{X_q, Y_q\}$ at q is

$$K_q(X,Y) = \frac{\langle R(Y,X)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}\Big|_q = (\frac{\partial \lambda}{\partial u} - \lambda^2)\Big|_{u=R},$$

It is possible to conclude K is a function of the base point only. By the Schur theorem, the sectional curvature is constant. □

Ruled Surfaces in space forms

Given $\psi(u,v)=\alpha(u)+v\,Z(u)$ in $\mathbb{E}^3$, then the Gaussian curvature satisfies

$$K=-\frac{(\psi_u,\psi_v,\psi_{uv})^2}{(\det g_{ij})^2}=-\left[\frac{(\alpha',Z,Z')}{\det g_{ij}}\right]^2.$$

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If Z is unit normal field along α, then $K = 0$ iff Z is RM.
In addition, $K = 0$ implies the surface is ruled.

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In addition, $K = 0$ implies the surface is ruled.

Theorem G [Portnoy, Pac. J. Math. 57 (1975) 281]

Let $\psi(u,v) = \exp_{\alpha(u)}(vZ(u))$ be a ruled surface in $\mathbb{H}^3$. Then,

$$K_{ext} = 0 \Leftrightarrow (\alpha', Z, \nabla_{\alpha'}Z) = 0.$$

In particular, if Z is unit normal field along α, then $K_{ext} = 0$ iff Z is RM.
In addition, $K_{ext} = 0$ implies the surface is ruled.

Extrinsic Gaussian curvature

Theorem 6 [— & da Silva, e-print arXiv:2106.10346 (2021)]

Let the generating curve α be the *striction curve*, $\langle\alpha',\nabla_{\alpha'}Z\rangle=0$, then

$$K_{ext}(u,v)=-\cos^2\left(\frac{v}{r}\right)\frac{\lambda(u)^2}{\left[\lambda(u)^2\cos^2\left(\frac{v}{r}\right)+r^2\sin^2\left(\frac{v}{r}\right)\right]^2},$$

where we have defined the *distribution parameter*

$$\lambda=\frac{(\alpha',Z,\nabla_{\alpha'}Z)}{\|\nabla_{\alpha'}Z\|^2}.$$

Note: For ruled surfaces in $\mathbb{H}^3(r)$, replace sin and cos by sinh and cosh.

Theorem 7 [— & da Silva, e-print arXiv:2106.10346 (2021)]

Every extrinsically flat surface in a space form is ruled.

Proof. Let $\Sigma : (u, v) \mapsto \psi(u, v)$ have $K_{ext} = 0$ and consider $\partial_1 = \frac{\partial}{\partial u}$, $\partial_2 = \frac{\partial}{\partial v}$, and $\xi = \partial_3$, where $\kappa_2(\partial_2) = 0$.

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$$\begin{aligned} \langle R(\partial_i, \partial_j)\partial_k, \xi\rangle_p &= \langle \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k, \xi\rangle_p \\ R_{ijk3}(p) &= h_{jm}\Gamma_{ik}^m - h_{im}\Gamma_{jk}^m + h_{ik,j} - h_{jk,i}\,. \end{aligned}$$

In a space form, $R_{ijk3} = K_0(\langle\partial_i, \partial_k\rangle\langle\partial_j, \partial_3\rangle - \langle\partial_i, \partial_3\rangle\langle\partial_j, \partial_k\rangle)$. Using that $h_{2i} = \langle\nabla_{\partial_2}\partial_i, \xi\rangle = -\langle\partial_i, \nabla_{\partial_2}\xi\rangle = 0$ implies $h_{21} = h_{22} = 0$, we have

$$h_{i1}\Gamma_{22}^1 = -R_{i223} = 0.$$

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$$h_{i1}\Gamma^1_{22} = -R_{i223} = 0.$$

At a parabolic point, $h_{11} \neq 0 \Rightarrow \Gamma^1_{22} = 0$ in a neighborhood of p. Finally,

$$\nabla_{\partial_2}\partial_2 = \Gamma^i_{22}\partial_i + h_{22}\xi = \Gamma^2_{22}\partial_2 \Rightarrow \kappa_g(\partial_2) \propto \langle\nabla_{\partial_2}\partial_2, -\partial_1\rangle = 0.$$

In conclusion, Σ is a union of pieces of rule surfaces. □

Characterization of space forms

Theorem 8 [— & da Silva, e-print arXiv:2106.10346 (2021)]

A manifold M^3 is a space form if and only if

1. there exists a surface with $K_{ext} = 0$ tangent to any 2d plane
2. if every surface with $K_{ext} = 0$ is ruled.

Proof. Given $\pi_p \subset T_pM^3$, let Σ^2 have $K_{ext} = 0$ and such that $T_p\Sigma^2 = \pi_p$. Write $\Sigma : (u, v) \mapsto \exp_{\alpha(u)}(vZ(u))$, where $\kappa_n(\partial_2) = 0$ and $\kappa_1 = \kappa_n(\partial_1)$.

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$$R_p(\partial_2, \partial_1)\partial_2 = \nabla_{\partial_1}\nabla_{\partial_2}\partial_2 - \nabla_{\partial_2}\nabla_{\partial_1}\partial_2 + \nabla_{[\partial_1,\partial_2]}\partial_2 = -\nabla_{\partial_2}\nabla_{\partial_1}\partial_2.$$

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Since $\langle\partial_2, \partial_2\rangle = 1 \Rightarrow \langle\nabla_{\partial_1}\partial_2, \partial_2\rangle = 0$, then $\nabla_{\partial_1}\partial_2 = A\partial_1 + B\xi$. Moreover,

$$\nabla_{\partial_2}\xi = 0 \Rightarrow B = \langle\xi, \nabla_{\partial_1}\partial_2\rangle = \langle\xi, \nabla_{\partial_2}\partial_1\rangle = \langle-\nabla_{\partial_2}\xi, \partial_1\rangle = 0$$

Finally, $\nabla_{\partial_1}\partial_2 = A\,\partial_1$ implies $K_p(\partial_1, \partial_2) = -(\frac{\partial A}{\partial v} + A^2)$ and, therefore, M^3 must be a space form. □

Theorem 9 [— & da Silva, e-print arXiv:2106.10346 (2021)]

Let $\Sigma(u,v) \subset M^3$ have $K_{ext} = 0$ and let the $\kappa_2(\partial_2) = 0$. The v-curves are geodesics, i.e., Σ^2 is ruled, if and only if $R_{1223} = 0$.

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Corollary 2 [– & da Silva, arXiv:2106.10346; Dillen et al, Monatsh. Math. 152 (2007) 89; Dillen & Munteanu, Bull. Braz. Math. Soc. 40 (2009) 85]

Every constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is a ruled surface.

Proof. Write $\partial_t = \sin\theta\, \mathrm{e}_2 + \cos\theta\, \mathrm{e}_3$, $\xi = \partial_3$, and $\mathrm{e}_1 = \mathrm{e}_2 \times \mathrm{e}_3$. Since

$$g_H(e_2^H, \mathrm{e}_1) = g(\mathrm{e}_2, \mathrm{e}_1) - g_R(e_2^R, 0) = 0 \text{ and}$$

$$g_H(\mathrm{e}_1, e_3^H) = g(\mathrm{e}_1, \mathrm{e}_3) - g_R(e_1^R, e_3^R) = 0,$$

then $R_{1223} = g_H(e_1, e_2^H) g_H(e_2^H, e_3^H) - g_H(e_1^H, e_3^H) g_H(e_2^H, e_2^H) = 0.$ □

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then $R_{1223} = g_H(e_1, e_2^H) g_H(e_2^H, e_3^H) - g_H(e_1^H, e_3^H) g_H(e_2^H, e_2^H) = 0.$ □

Theorem 10 [— & da Silva, e-print arXiv:2106.10346 (2021)]

There exist Σ^2 in $\mathbb{H}^2\times\mathbb{R}$ and $\mathbb{S}^2\times\mathbb{R}$ with $K_{ext}=0$ which does not make a constant angle. In addition, the asymptotic directions are orthogonal to ∂_t.

Summary

- Several properties of spherical curves in space forms are valid because their spheres are totally umbilical.
- We we force those properties to be valid for spherical curves, then we are forced to work with space forms.
- We proved, using RM frames, that umbilical spheres is a characteristic property of space forms.
- We provided a necessary and sufficient condition for an extrinsically flat surface in a generic manifold be ruled.
- Space forms are the only manifolds which have plenty of extrinsically flat surfaces and they are all ruled.
- We showed that constant angle surfaces in homogeneous product manifolds are ruled.
- There must exist extrinsically flat surfaces in homogeneous product manifolds which do not make a constant angle with the real direction.

Muito obrigado! (Thank you!)

Our works

1. Da Silva, L. C. B., Da Silva, J. D.: *Characterization of curves that lie on a geodesic sphere or on a totally geodesic hypersurface in a hyperbolic space or in a sphere*. Mediterr. J. Math. **15** (2018) 70.
2. Da Silva, L. C. B., Da Silva, J. D.: *Characterization of manifolds of constant curvature by spherical curves*. Annali di Matematica **199** (2020) 217–229.
3. Da Silva, L. C. B., Da Silva, J. D.: *Characterization of manifolds of constant curvature by ruled surfaces*. e-print arXiv:2106.10346.

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