On boundary decay of harmonic functions, Green kernels and heat kernels for some non-local operators

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Supported in part by the Croatian Science Foundation under the project 4197

8ECM Portorož
Nonlocal operators and related topics
June 24, 2021

Fractional Laplacian in an open set

Let $D \subset \mathbb{R}^d$ be an open set, and $\beta \in (0, 2]$. Consider β-stable process killed upon exiting D. The corresponding Dirichlet form (in case $\beta \in (0, 2)$) is given by

$$\mathcal{E}(u, u) = \int_D \int_D (u(y) - u(x))^2 |x - y|^{-d-\beta}\, dx\, dy + \int_D u(x)^2 \kappa(x)\, dx,$$

where

$$\kappa(x) = \int_{D^c} |x - y|^{-d-\beta}\, dy \asymp \delta_D(x)^{-\beta}$$

(when D is $C^{1,1}$) is the killing potential.

The infinitesimal generator of the semigroup is $\beta/2$-fractional Laplacian with zero exterior condition written as $(-\Delta)^{\beta/2}|_D$ – the restricted fractional Laplacian (RFL).

The sharp two-sided heat kernel estimates in $C^{1,1}$-open set D were established by Chen, Kim, Song (JEMS 2010) (for $\beta \in (0, 2)$):

$$p_D(t, x, y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\beta}}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\beta}}\right)^{\beta/2} \left(t^{-d/\beta} \wedge \frac{t}{|x - y|^{d+\beta}}\right)$$

for small time t ($|x - y|^\beta \leq t$ - near-diagonal), and (if D is bounded)

$$p_D(t, x, y) \asymp e^{-\lambda_1 t} \delta_D(x)^{\beta/2} \delta_D(y)^{\beta/2}$$

for large time t (λ_1 the first eigenvalue of $(-\Delta)^{\beta/2}|_D$).

By integrating $p_D(t,x,y)$ over time, one gets the sharp two-sided Green function estimates

$$G_D(x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\beta/2} |x-y|^{-d+\beta}$$

(Chen, Song 1998; Kulczycky 1997).
The boundary Harnack principle holds for non-negative harmonic functions with the exact decay rate $\delta_D(x)^{\beta/2}$ (Bogdan 1997).

One can regard the RFL as a Schrödinger perturbation of the regional Laplacian in D:

$$Lu(x) = \text{p.v.} \int_D (u(y)-u(x))|x-y|^{-d-\beta},$$

namely, $(-\Delta)^{\beta/2}|_D u(x) = Lu(x) - \kappa(x)u(x)$ with critical perturbation $\kappa(x) \asymp \delta_D(x)^{-\beta}$.
It was shown by Chen, Kim, Song (PTRF 2010) that for $\beta \in (1,2)$ the censored process corresponding to the regional Laplacian has the heat kernel estimates

$$q_D(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\beta}}\right)^{\beta-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\beta}}\right)^{\beta-1} \left(t^{-d/\beta} \wedge \frac{t}{|x-y|^{d+\beta}}\right)$$

For subcritical perturbation κ, for example $\kappa(x) = c\delta_D(x)^{-\rho}$ with $0 \le \rho < \beta$ there is stability of the heat kernel, cf. Chen, Kim, Song, (TAMS 2015).

In Cho, Kim, Song, V., *Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings* (JMPA 2020), in case $\beta \in (1,2)$, we considered critical perturbations of the regional Laplacian of the form $\kappa(x) = C(\beta, p)\delta_D(x)^{-\beta}$ where $C(\beta, p) \in [0, \infty)$ is a constant depending on the parameter $p \in [\beta - 1, \beta)$, $\lim_{p \downarrow \beta - 1} C(\beta, p) = 0$, $\lim_{p \uparrow \beta} C(\beta, p) = \infty$.
The sharp two-sided heat kernel estimates are

$$q(t, x, y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\beta}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\beta}}\right)^p \left(t^{-d/\beta} \wedge \frac{t}{|x-y|^{d+\beta}}\right).$$

Note that $p = \beta - 1$ corresponds to the regional fractional Laplacian, while $p = \beta/2$ corresponds to the restricted fractional Laplacian.
Also note that the boundary decay rate $p \in [\beta - 1, \beta)$ depends on the constant $C(\beta, p)$.

Fractional power of the RFL

For $\gamma \in (0,1)$ consider the fractional power $((-\Delta)^{\beta/2}|_D)^\gamma$. When $\beta = 2$, this is the usual spectral fractional Laplacian (SFL). Probabilistically, one subordinates the killed β-stable process by means of an independent γ-stable subordinator.
Some aspects of this operator (in case of an open $C^{1,1}$ set), in particular which properties depend on β and which on γ, were studied by Kim, Song, V in TAMS (2019, $\beta = 2$) and Pot. Anal. (2020, $\beta \in (0,2)$).
Let $\alpha := \beta\gamma \in (0,2)$. The sharp two-sided Green function estimates are

$$
\begin{aligned}
G(x,y) &\asymp \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\beta/2} |x-y|^{-d+\alpha} \\
&= \left(1 \wedge \frac{\delta_D(x) \wedge \delta_D(y)}{|x-y|}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(x) \vee \delta_D(y)}{|x-y|}\right)^{\beta/2} |x-y|^{-d+\alpha}
\end{aligned}
$$

The Dirichlet form is given by

$$
\mathcal{E}(u,u) = \int_D \int_D (u(y)-u(x))^2 J(x,y)\,dy\,dx + \int_D u(x)^2 \kappa(x)\,dx
$$

with $\kappa(x) \asymp \delta_D(x)^{-\alpha}$.
The most interesting ingredient is the (jump) kernel $J(x,y)$ which has a rather unusual two-sided estimates:

In case $\beta = 2$

$$J(x,y) \asymp \left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right) \left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right) |x-y|^{-d-\alpha}$$

(does not depend on γ).

In case $\beta = 2$

$$J(x,y) \asymp \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D(x)\vee\delta_D(y)}{|x-y|}\wedge 1\right)|x-y|^{-d-\alpha}$$

(does not depend on γ).
Somewhat surprisingly, in case $\beta \in (0,2)$

$$J(x,y) \asymp \begin{cases} \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\wedge 1\right)^{\beta(1-\gamma)}|x-y|^{-d-\alpha}, & \gamma\in(1/2,1),\\ \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\wedge 1\right)^{\beta/2}\log\left(1+\left(\frac{(\delta_D(x)\vee\delta_D(y))\wedge|x-y|}{(\delta_D(x)\wedge\delta_D(y))\wedge|x-y|}\right)^{\beta}\right)|x-y|^{-d-\alpha}, & \gamma=1/2,\\ \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\wedge 1\right)^{\beta/2}\left(\frac{\delta_D(x)\vee\delta_D(y)}{|x-y|}\wedge 1\right)^{(\beta/2)(1-2\gamma)}|x-y|^{-d-\alpha}, & \gamma\in(0,1/2). \end{cases}$$

If we write $J(x,y) = \mathcal{B}(x,y)|x-y|^{-d-\alpha}$, then the red part above is comparable to $\mathcal{B}(x,y)$. We call $\mathcal{B}(x,y)$ the boundary part of $J(x,y)$.

Another very surprising fact is that in case $\beta \in (0,2)$ the BHP holds when $\gamma \in (1/2,1)$, while it fails for $\gamma \in (0,1/2]$ (although Carleson's estimate holds true). In case $\beta = 2$, BHP holds for all $\gamma \in (0,1)$. When true, the decay rate of harmonic functions is $\delta_D(x)^{\beta/2}$ (independent of γ).

The heat kernel of the fractional power of the RFL

The semigroup of $((-\Delta)^{\beta/2}|_D)^\gamma$ has a density (the heat kernel) given by

$$q(t,x,y) = \int_0^\infty p_D(s,x,y)\mathbb{P}(S_t \in ds)$$

where $p_D(s,x,y)$ is the heat kernel of the RFL, and $S=(S_t)_{t\ge 0}$ is the γ-stable subordinator. To estimate the above integral one needs sharp estimates of densities $\mathbb{P}(S_t \in ds)$. The key estimate is $\mathbb{P}(S_t \ge s) \asymp ts^{-\gamma}$ for $t < s^\gamma$.

The sharp two-sided estimates (for bounded $C^{1,1}$-open set D) are established in Cho, Kim, Song, V: *Heat kernel estimates for subordinate Markov processes and their applications* (2021).
Suppose $t \in (0,1]$ (small time).
Near-diagonal estimates: if $|x-y|^\alpha \le t$, then

$$q(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\beta/2} t^{-d/\alpha}$$

Off-diagonal estimates: $|x-y|^\alpha > t$ and $\beta = 2$ (γ-SFL):

$$q(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \frac{t}{|x-y|^{d+2\gamma}}.$$

$$\delta_\vee(x,y) = \delta_D(x) \vee \delta_D(y), \qquad \delta_\wedge(x,y) = \delta_D(x) \wedge \delta_D(y),$$
$$\mathbf{m}_\vee(t,x,y) = (t^{1/\alpha} \vee \delta_\vee(x,y)) \wedge |x-y|, \qquad \mathbf{m}_\wedge(t,x,y) = (t^{1/\alpha} \vee \delta_\wedge(x,y)) \wedge |x-y|.$$

Off-diagonal estimates: $|x-y|^\alpha > t$ and $\beta \in (0,2)$:

$$q(t,x,y) \asymp \begin{cases} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\beta/2} \left(\frac{\mathbf{m}_\wedge(t,x,y)}{|x-y|}\right)^{\beta(1-\gamma)} \frac{t}{|x-y|^{d+\alpha}}, & \gamma > 1/2, \\ \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\beta/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\beta/2} \left(\frac{\mathbf{m}_\vee(t,x,y)}{|x-y|}\right)^{-\alpha} \frac{t}{|x-y|^{d+\alpha}}, & \gamma < 1/2, \\ \left(1 \wedge \frac{\delta_\wedge(x,y)}{|x-y|}\right)^{\beta/2} \left(1 \wedge \frac{\delta_\vee(x,y)}{t^{1/\alpha}}\right)^{\beta/2} \log\left(e + \frac{\mathbf{m}_\vee(t,x,y)}{\mathbf{m}_\wedge(t,x,y)}\right) \frac{t}{|x-y|^{d+\alpha}}, & \gamma = 1/2. \end{cases}$$

Suppose $t \geq 1$ (large time):

$$q(t,x,y) \asymp e^{-t\lambda_1^\gamma} \delta_D(x)^{\beta/2} \delta_D(y)^{\beta/2},$$

where λ_1 is the smallest eigenvalue of the Dirichlet (fractional) Laplacian $(-\Delta)^{\beta/2}|_D$.

The Dirichlet form degenerate at the boundary

The state space is the upper half-space $\mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : x_d > 0\}$. Let $\alpha \in (0, 2)$. The jump kernel:

$$J(x, y) = |x - y|^{-d-\alpha}\mathcal{B}(x, y) \text{ on } \mathbb{R}^d_+ \times \mathbb{R}^d_+$$

In case $0 < c \le \mathcal{B}(x, y) \le C$, this is well studied and can be viewed as a uniform elliptic condition for non-local operator (fractional Laplacian). One introduces the pure-jump Dirichlet form

$$\mathcal{E}(u, u) = \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x) - u(y))^2 J(x, y)\, dy\, dx + \int_{\mathbb{R}^d_+} u(x)^2\kappa(x)\, dx$$

and shows that there is a corresponding Hunt process Y which is Feller and strongly Feller.

Motivated by the jump kernel of $((-\Delta)^{\beta/2}|_{\mathbb{R}^d_+})^\gamma$, we try to develop the theory where $\mathcal{B}(x, y)$ depends on $x_d = \delta_{\mathbb{R}^d_+}(x)$, $y_d = \delta_{\mathbb{R}^d_+}(y)$, as well as $|x - y|$, and decays at the boundary.

The jump kernel

Kim, Song, V: *On potential theory of Markov processes with jump kernels decaying at the boundary* (2020)
Kim, Song, V: *Sharp two-sided Green function estimates for Dirichlet forms degenerate at the boundary* (2021)

Assumptions on the boundary function $\mathcal{B}(x,y)$:

(A1) $\mathcal{B}(x,y)=\mathcal{B}(y,x)$ for all $x,y\in\mathbb{R}^d_+$.
(A2) If $\alpha\ge 1$, then there exist $\theta>\alpha-1$ and $C>0$ such that

$$|\mathcal{B}(x,x)-\mathcal{B}(x,y)|\le C\left(\frac{|x-y|}{x_d\wedge y_d}\right)^{\theta}.$$

(A4) Scaling: For all $x,y\in\mathbb{R}^d_+$ and $a>0$, $\mathcal{B}(ax,ay)=\mathcal{B}(x,y)$.
Horizontal translation invariance: In case $d\ge 2$, for all $x,y\in\mathbb{R}^d_+$ and $\widetilde{z}\in\mathbb{R}^{d-1}$,

$$\mathcal{B}(x+(\widetilde{z},0),y+(\widetilde{z},0))=\mathcal{B}(x,y).$$

The jump kernel, cont.

(A3) There exist $C \geq 1$ and parameters $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C^{-1}\widetilde{B}(x,y) \leq \mathcal{B}(x,y) \leq C\widetilde{B}(x,y), \qquad x, y \in \mathbb{R}^d_+,$$

where

$$\begin{aligned}\widetilde{B}(x,y) \;&:=\; \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{\beta_2} \\ &\times \left[\log\left(1 + \frac{(x_d \vee y_d) \wedge |x-y|}{x_d \wedge y_d \wedge |x-y|}\right)\right]^{\beta_3} \\ &\times \left[\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right]^{\beta_4}.\end{aligned}$$

$$J(x,y) \asymp \begin{cases} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta(1-\gamma)} |x-y|^{-d-\alpha}, & \gamma \in (1/2, 1), \\ \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta/2} \log\left(1 + \left(\frac{(x_d \vee y_d) \wedge |x-y|}{(x_d \wedge y_d) \wedge |x-y|}\right)^{\beta}\right) |x-y|^{-d-\alpha}, & \gamma = 1/2, \\ \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta/2} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{(\beta/2)(1-2\gamma)} |x-y|^{-d-\alpha}, & \gamma \in (0, 1/2). \end{cases}$$

The killing potential

To every parameter $p \in ((\alpha-1)_+, \alpha+\beta_1)$, we associate a constant $C(\alpha, p, \mathcal{B}) \in (0, \infty)$ depending on α, p and $\mathcal{B}$ defined as

$$C(\alpha, p, \mathcal{B}) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^p-1)(1-s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}\big((1-s)\widetilde{u}, 1), s\mathbf{e}_d\big)\, ds\, d\widetilde{u}\,, \quad (1)$$

where $\mathbf{e}_d = (\tilde{0}, 1)$.
The function $p \mapsto C(\alpha, p, \mathcal{B})$ is strictly increasing, continuous, and

$$\lim_{p \downarrow (\alpha-1)_+} C(\alpha, p, \mathcal{B}) = 0, \qquad \lim_{p \uparrow \alpha+\beta_1} C(\alpha, p, \mathcal{B}) = \infty.$$

The killing potential is defined by

$$\kappa(x) = C(\alpha, p, \mathcal{B}) x_d^{-\alpha}, \qquad x \in \mathbb{R}^d_+.$$

Dirichlet form

Let

$$\mathcal{E}^{\mathbb{R}^d_+}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx,$$

and let $\mathcal{F}^{\mathbb{R}^d_+}$ be the closure of $C_c^\infty(\mathbb{R}^d_+)$ under $\mathcal{E}_1^{\mathbb{R}^d_+} := \mathcal{E}^{\mathbb{R}^d_+} + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}$.
Then $(\mathcal{E}^{\mathbb{R}^d_+}, \mathcal{F}^{\mathbb{R}^d_+})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$.

Set

$$\mathcal{E}(u,v) := \mathcal{E}^{\mathbb{R}^d_+}(u,v) + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa(x)\,dx\,,$$

and $\mathcal{F} = \widetilde{\mathcal{F}^{\mathbb{R}^d_+}} \cap L^2(\mathbb{R}^d_+, \kappa(x)dx)$.
Then $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$.

The operator

For $u : \mathbb{R}^d_+ \to \mathbb{R}$ and $x \in \mathbb{R}^d_+$, we set

$$L^{\mathcal{B}}_\alpha u(x) := \text{p.v.} \int_{\mathbb{R}^d_+} (u(y) - u(x)) J(x, y)\, dy = \text{p.v.} \int_{\mathbb{R}^d_+} \frac{u(y) - u(x)}{|x - y|^{d+\alpha}} \mathcal{B}(x, y),$$

whenever the principal value integral makes sense. Further, let

$$L^{\mathcal{B}} u(x) := L^{\mathcal{B}}_\alpha u(x) - \kappa(x) u(x) = L^{\mathcal{B}}_\alpha u(x) - C(\alpha, p, \mathcal{B}) x_d^{-\alpha} u(x), \quad x \in \mathbb{R}^d_+.$$

Then (at least formally), $\mathcal{E}(u, v) = (-L^{\mathcal{B}} u, v)_{L^2(\mathbb{R}^d_+)}$, i.e., $L^{\mathcal{B}}$ is the generator of the corresponding semigroup (and the process).

Explanation of $C(\alpha, p, \mathcal{B})$: If $g_p(y) := y_d^p$, then

$$L^{\mathcal{B}}_\alpha g_p(x) = C(\alpha, p, \mathcal{B}) x_d^{p-\alpha}.$$

The process

Let $((Y_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d_+\setminus\mathcal{N}})$ be the associated Hunt process with lifetime ζ. It can be proved that the exceptional set $\mathcal{N}$ can be taken as the empty set. We add a cemetery point ∂ to the state space $\mathbb{R}^d_+$ and define $Y_t = \partial$ for $t \geq \zeta$.

A special case: $\mathcal{B}(x,y) = 1$ – no boundary term, and $p = \alpha/2$. Then Y is the isotropic α-stable case killed upon exiting $\mathbb{R}^d_+$. Recall the Green function estimates: on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p$$

No boundary term, no killing – not part of the setting: When $\alpha \in (1,2)$ the corresponding process is the censored α-stable process. The Green function estimates as above with $p = \alpha - 1$.

Green function

Let Y be the Hunt process with lifetime ζ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. For a measurable function $f : \mathbb{R}^d_+ \to [0, \infty)$ define the Green potential by

$$Gf(x) := \mathbb{E}_x \int_0^\zeta f(Y_t)\, dt = \int_0^\infty P_t f(x)\, dt, \quad x \in \mathbb{R}^d_+.$$

Under the assumption **(A1)** and **(A2)** one can show that there exists a symmetric function $G(x, y)$ (excessive in both variables) such that

$$Gf(x) = \int_{\mathbb{R}^d_+} G(x, y) f(y)\, dy.$$

The function $G(x, y)$ is called the Green function.

As a consequence of scaling **(A4)** we have that

$$G(x, y) = G\left(\frac{x}{|x-y|}, \frac{y}{|x-y|}\right) |x-y|^{\alpha-d}, \quad x, y \in \mathbb{R}^d_+.$$

Green function estimates

Theorem A: Assume **(A1)– (A4)** hold, $\kappa(x) = C(\alpha, p, \mathcal{B})x_d^{-\alpha}$, $p \in ((\alpha-1)_+, \alpha+\beta_1)$ and $d > 2 \wedge (\alpha+\beta_1+\beta_2)$.
Then the process Y admits a Green function $G : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to [0, \infty]$ such that $G(x, \cdot)$ is continuous in $\mathbb{R}^d_+ \setminus \{x\}$ and regular harmonic with respect to Y in $\mathbb{R}^d_+ \setminus B(x, \epsilon)$ for any $\epsilon > 0$.
Moreover, $G(x, y)$ has the following estimates:

(1) If $p \in ((\alpha-1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p = \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^p \frac{1}{|x-y|^{d-}}$$

(2) If $p = \alpha + \frac{\beta_1+\beta_2}{2}$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p \left(\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right)^{\beta_4+1}.$$

(3) If $p \in (\alpha + \frac{\beta_1+\beta_2}{2}, \alpha + \beta_1)$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$\begin{aligned} G(x,y) &\asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{2\alpha-p+\beta_1+\beta_2} \log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)^{\beta} \\ &= \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{-2(p-\alpha-(\beta_1+\beta_2)/2)} \\ &\quad \times \left(\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right)^{\beta_4}. \end{aligned}$$

Green potential

For any $a, b > 0$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, define a box
$$D_{\widetilde{w}}(a,b) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x} - \widetilde{w}| < a, 0 < x_d < b\}.$$

Proposition: For any $\widetilde{w} \in \mathbb{R}^{d-1}$, any Borel set D satisfying $D_{\widetilde{w}}(R/2, R/2) \subset D \subset D_{\widetilde{w}}(R, R)$ and any $x = (\widetilde{w}, x_d)$ with $x_d \le R/10$ (comparison constant independent of $\widetilde{w}$, D, R and x)

$$\mathbb{E}_x \int_0^{\tau_D} (Y_t^d)^\beta \, dt = \int_D G^D(x,y) y_d^\beta \, dy \asymp \begin{cases} R^{\alpha+\beta-p} x_d^p, & \beta > p - \alpha, \\ x_d^p \log(R/x_d), & \beta = p - \alpha, \\ x_d^{\alpha+\beta}, & -p-1 < \beta < p - \alpha, \\ \infty, & \beta \le -p-1. \end{cases} \tag{2}$$

Corollary: For all $x \in \mathbb{R}^d_+$,
$$\mathbb{E}_x \int_0^{\zeta} (Y_t^d)^\beta \, dt = \int_{\mathbb{R}^d_+} G(x,y) y_d^\beta dy \asymp \begin{cases} \infty & \beta \ge p - \alpha \text{ or } \beta \le -p-1, \\ x_d^{\alpha+\beta}, & -p-1 < \beta < p - \alpha. \end{cases}$$

For similar results cf. Abatangelo, Gomez-Castro, Vazquez: *Singular boundary behaviour and large solutions for fractional elliptic equations* (2019).

Boundary Harnack principle

For any $a, b > 0$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, define a box
$$D_{\widetilde{w}}(a,b) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x} - \widetilde{w}| < a, 0 < x_d < b\}.$$

Theorem B: Assume $p \in ((\alpha-1)_+, \alpha + (\beta_1 \wedge \beta_2))$. Then there exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have
$$\frac{f(x)}{x_d^p} \leq C \frac{f(y)}{y_d^p}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2). \tag{3}$$

A consequence is that if two functions f, g in $\mathbb{R}^d_+$ both satisfy the assumptions in Theorem B, then
$$\frac{f(x)}{f(y)} \leq C^2 \frac{g(x)}{g(y)}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Failure of the BHP

Theorem C: Assume $\alpha + \beta_2 \le p < \alpha + \beta_1$. Then the non-scale-invariant boundary Harnack principle is not valid for Y.

The non-scale-invariant boundary Harnack principle holds near the boundary of $\mathbb{R}^d_+$ if there is a constant $\widehat{R} \in (0,1)$ such that for any $r \in (0, \widehat{R}\,]$, there exists a constant $c = c(r) \ge 1$ such that for all $\widetilde{w} \in \mathbb{R}^{d-1}$ and non-negative functions f, g in $\mathbb{R}^d_+$ which are harmonic in $\mathbb{R}^d_+ \cap B((\widetilde{w}, 0), r)$ with respect to Y and vanish continuously on $\partial\mathbb{R}^d_+ \cap B((\widetilde{w}, 0), r)$, we have

$$\frac{f(x)}{f(y)} \le c\,\frac{g(x)}{g(y)} \qquad \text{for all } x, y \in B((\widetilde{w}, 0), r/2) \cap \mathbb{R}^d_+.$$

Thank you.