Bipartite 2-factorizations of complete multigraphs via layering

Mateja Šajna

University of Ottawa

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Joint work with Amin Bahmanian

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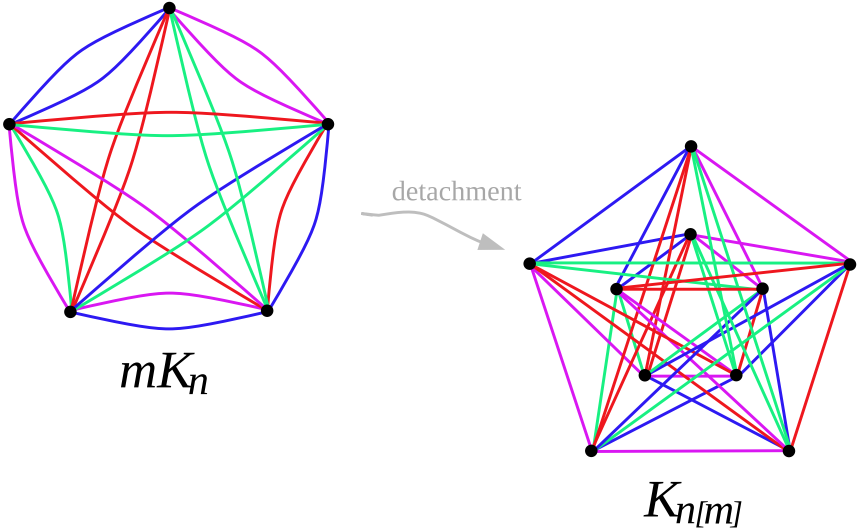
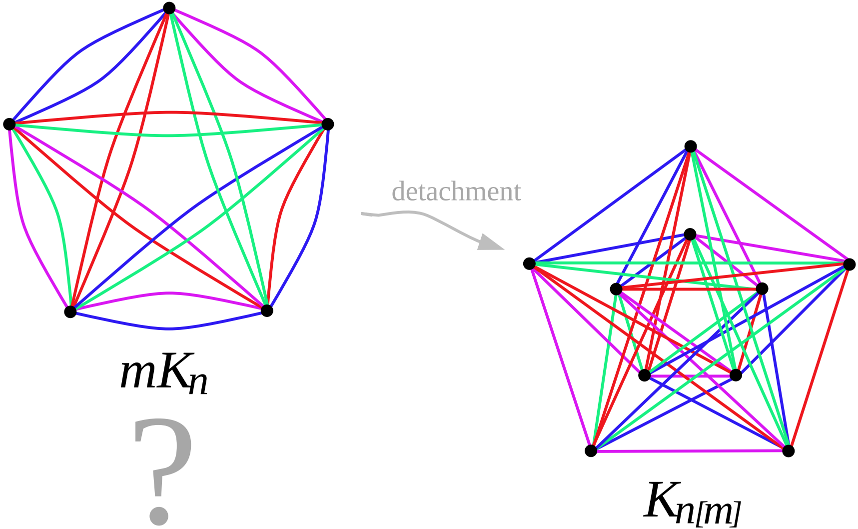
Outline

- Motivation
- Terminology
- Layering — a simple idea
- The layering theorem
- Main result: bipartite 2-factorizations of λK_n
- Corollary: the Oberwolfach Problem
- Corollary: the Hamilton-Waterloo Problem

Motivation: 2-factorizations of complete equipartite multigraphs via detachment



Motivation: 2-factorizations of complete equipartite multigraphs via detachment



Terminology: multisets

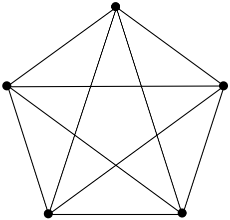
- Multiset with $\mathrm{mult}_M(x)$ copies of x: $M = [x^{\langle \mathrm{mult}(x) \rangle} : x \in \mathcal{U}]$
- E.g. $[1, 1, 3, 3, 4, 7, 8] = [1^{\langle 2 \rangle}, 3^{\langle 2 \rangle}, 4, 7, 8]$

- Multiset union: $M_1 \sqcup M_2 = [x^{\langle \mathrm{mult}_{M_1}(x) + \mathrm{mult}_{M_2}(x) \rangle} : x \in \mathcal{U}]$
- E.g. $[1^{\langle 2 \rangle}, 3^{\langle 2 \rangle}, 4, 7, 8] \sqcup [1^{\langle 2 \rangle}, 3, 7^{\langle 2 \rangle}, 8] = [1^{\langle 4 \rangle}, 3^{\langle 3 \rangle}, 4, 7^{\langle 3 \rangle}, 8^{\langle 2 \rangle}]$

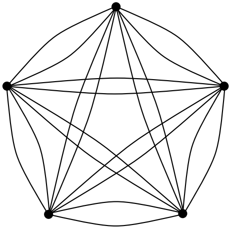
- Refinement of $M = [x_1, x_2, \ldots, x_s]$: $M' = [y_1, y_2, \ldots, y_t]$ such that there exists a partition $\{P_1, P_2, \ldots, P_s\}$ of $\{1, 2, \ldots, t\}$ with $\sum_{j \in P_i} y_j = x_i$ for $i = 1, 2, \ldots, s$
- E.g. $[1, 1, 1, 2, 3, 3, 4, 5, 7, 8]$ is a refinement of $[1, 1, 3, 7, 7, 8, 8]$

Terminology: multigraphs

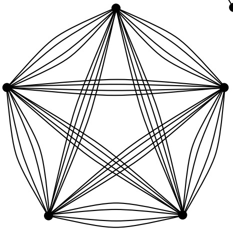
- $\lambda\mathcal{G}$: λ-fold graph $\mathcal{G}$



K_n



$2K_n$



$4K_n = 2(2K_n)$

Terminology: factors

- r-factor of $\mathcal{G}$: spanning r-regular subgraph of $\mathcal{G}$
- Type of 2-factor F: $[c_1, \ldots, c_t]$
  if F is a disjoint union of cycles of lengths $c_1, \ldots, c_t$
- Bipartite 2-factor type: $T = [c_1, \ldots, c_t]$ with c_i all even
- 2-factor type admissible for $\mathcal{G}$: $\mathcal{G}$ contains a 2-factor of this type

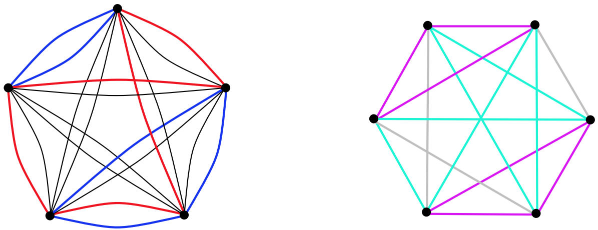


Figure: 2-factors of types $[2, 3]$ and $[5]$ in $2K_5$ (left); a 1-factor and 2-factors of types $[3, 3]$ and $[6]$ in K_6 (right).

Terminology: 2-factorizations

- 2-factorization of $\mathcal{G}$:
  - decomposition of $\mathcal{G}$ into 2-factors, if $\mathcal{G}$ is $2k$-regular
  - decomposition of $\mathcal{G}$ into 2-factors and a 1-factor, if $\mathcal{G}$ is $(2k+1)$-regular
- Type of 2-factorization $\{F_1, \ldots, F_k\}$: $[T_1, \ldots, T_t]$ if 2-factor F_i is of type T_i, for all i
- Bipartite 2-factorization type: $\mathcal{T} = [T_1, \ldots, T_k]$ with T_i all bipartite 2-factor types
- 2-factorization type admissible for $\mathcal{G}$: $\mathcal{T} = [T_1, \ldots, T_k]$ with T_i all admissible for $\mathcal{G}$, and $\mathcal{G}$ is $2k$-regular or $(2k+1)$-regular

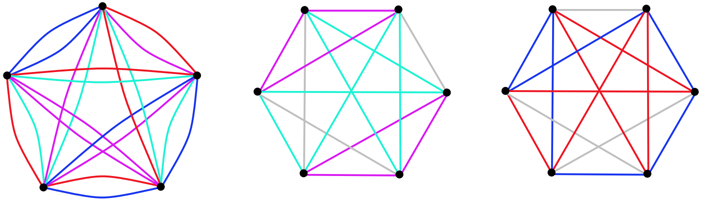


Figure: 2-factorization of type $[[2,3],[2,3],[5],[5]]$ in $2K_5$ (left); 2-factorizations of types $[[3,3],[6]]$ and $[[6],[6]]$ in K_6 (centre and right).

Example: $K_5 \oplus K_5 = 2K_5$

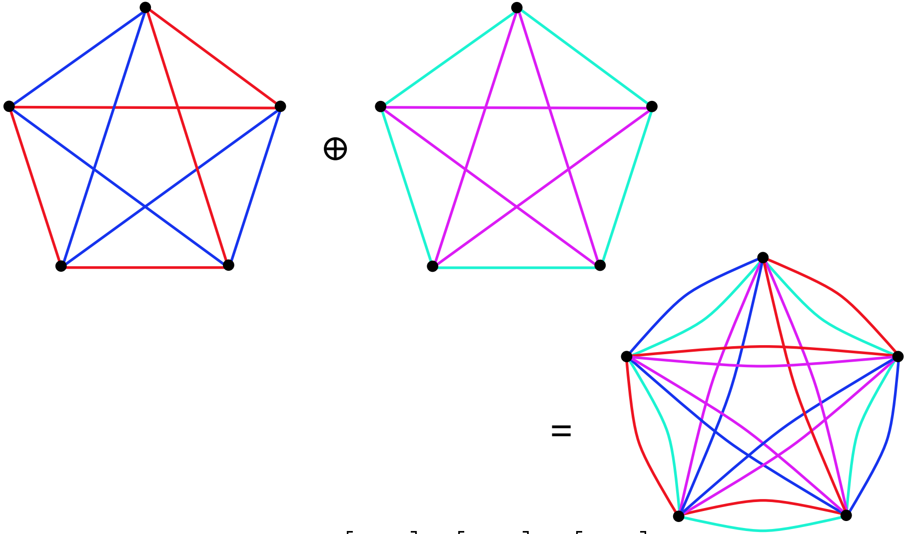


Figure: $\left[[5]^{\langle 2\rangle}\right] \sqcup \left[[5]^{\langle 2\rangle}\right] = \left[[5]^{\langle 4\rangle}\right]$

Example: $K_6 \oplus 2K_6 = 3K_6$

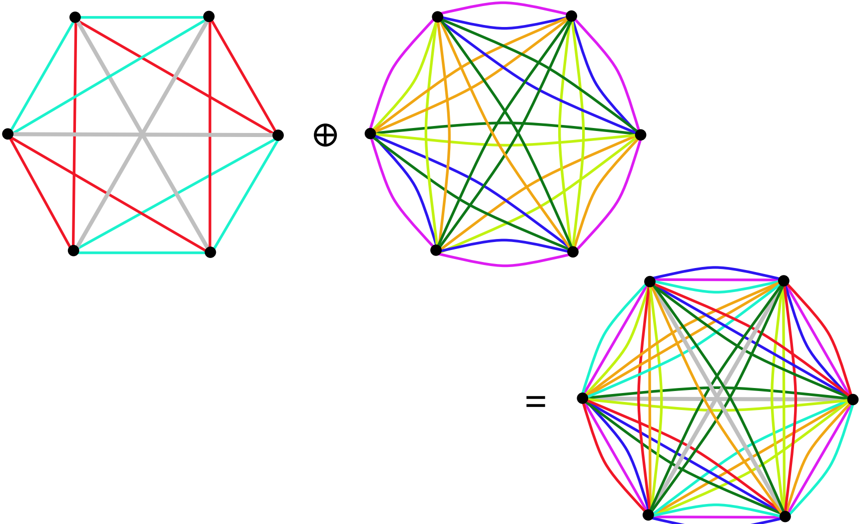


Figure: $\left[[3^{\langle 2\rangle}],[6]\right] \sqcup \left[[2,4]^{\langle 3\rangle},[3^{\langle 2\rangle}],[6]\right] = \left[[2,4]^{\langle 3\rangle},[3^{\langle 2\rangle}]^{\langle 2\rangle},[6]^{\langle 2\rangle}\right]$

Example: $K_6 \oplus K_6 = 2K_6$

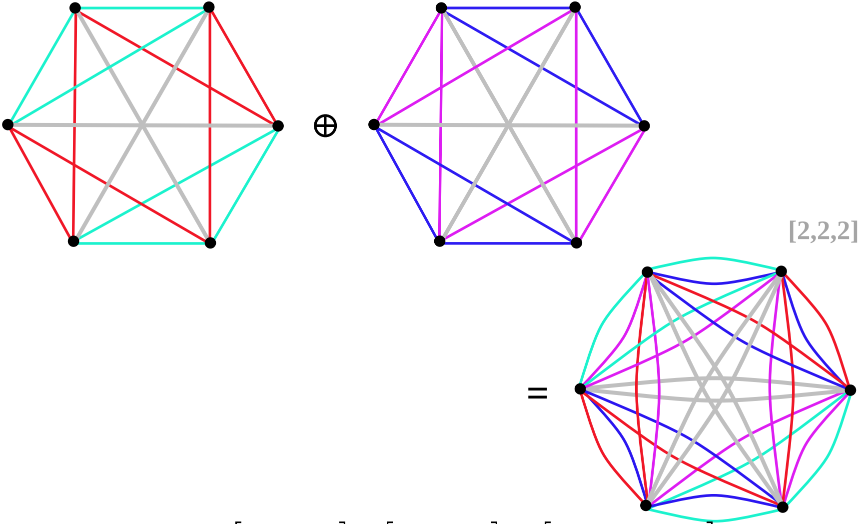


Figure: $\left[[3^{\langle 2\rangle}],[6]\right] \sqcup \left[[3^{\langle 2\rangle}],[6]\right] = \left[[3^{\langle 2\rangle}]^{\langle 2\rangle},[6]^{\langle 2\rangle}\right]$

Example: $K_6 \oplus K_6 = 2K_6$

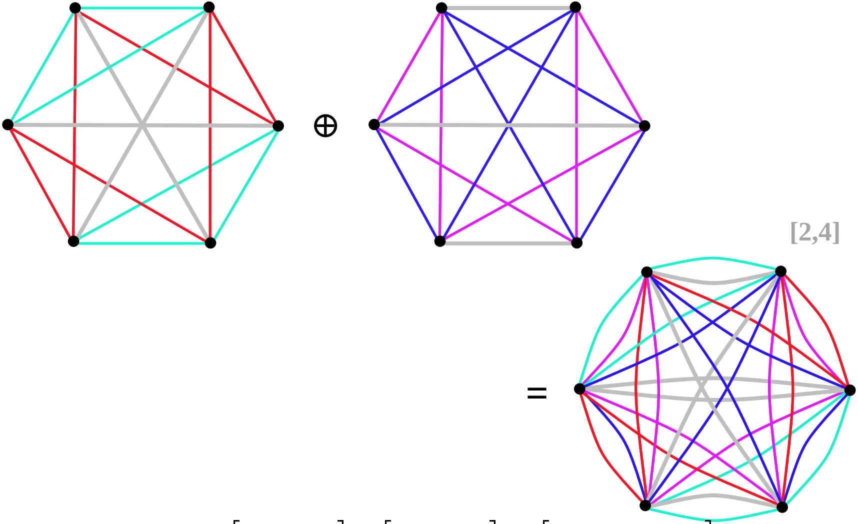


Figure: $\left[[3^{\langle 2 \rangle}], [6]\right] \sqcup \left[[3^{\langle 2 \rangle}], [6]\right] = \left[[3^{\langle 2 \rangle}]^{\langle 2 \rangle}, [6]^{\langle 2 \rangle}\right]$

Example: $K_6 \oplus K_6 = 2K_6$

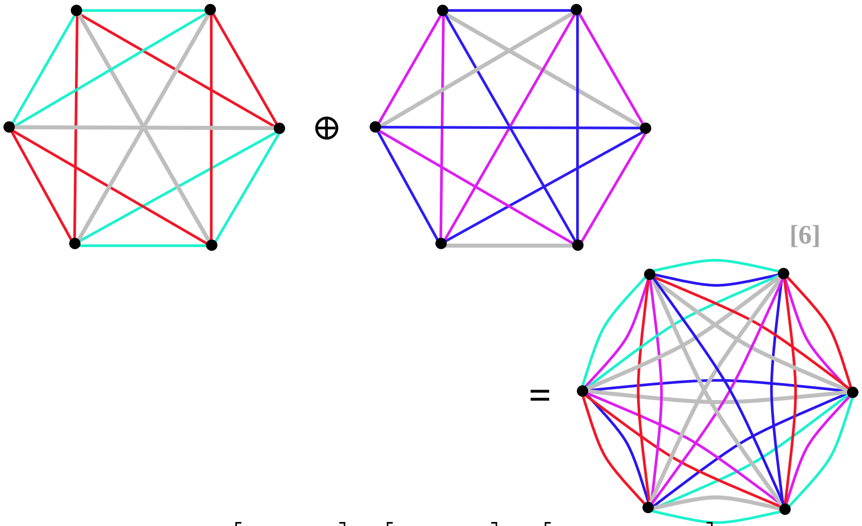


Figure: $\left[[3^{\langle 2 \rangle}], [6]\right] \sqcup \left[[3^{\langle 2 \rangle}], [6]\right] = \left[[3^{\langle 2 \rangle}]^{\langle 2 \rangle}, [6]^{\langle 2 \rangle}\right]$

The layering theorem

Theorem 1 (Bahmanian and Šajna, 2021^+)

For $i = 1, \ldots, \ell$, *assume* $\mu_i K_n$ *admits a 2-factorization of type* $\mathcal{T}_i = [T_{i,1}, \ldots, T_{i,k_i}]$, *with* $k_i = \lfloor \frac{\mu_i(n-1)}{2} \rfloor$.
Let $\mu = \sum_{i=1}^{\ell} \mu_i$ *and* $\beta = |\{i : \mu_i(n-1)$ *is odd*$\}|$.

- *Then* μK_n *admits a 2-factorization of type*

$$\sqcup_{i=1}^{\ell}[T_{i,1}, \ldots, T_{i,k_i}] \sqcup [T_1, \ldots, T_{\lfloor \frac{\beta}{2} \rfloor}]$$

  for all bipartite 2-factor types $T_1, \ldots, T_{\lfloor \frac{\beta}{2} \rfloor}$ *that are admissible for* $2K_n$.

Bipartite 2-factorizations of K_n: previous results

Theorem 2 (Häggkvist, 1985)

Let $n \equiv 2 \pmod 4$, *and let* $\mathcal{T} = [T_1^{\langle \alpha_1 \rangle}, \ldots, T_k^{\langle \alpha_k \rangle}]$ *be any admissible bipartite 2-factorization type with all* α_i *even.*

- *Then* K_n *admits a 2-factorization of type* $\mathcal{T}$*.*

Theorem 3 (Bryant and Danziger, 2011)

Let $n \equiv 0 \pmod 4$, *and let* $\mathcal{T} = [T_1^{\langle \alpha_1 \rangle}, \ldots, T_k^{\langle \alpha_k \rangle}]$ *be any admissible bipartite 2-factorization type with* $\alpha_1 \geq 3$ *odd, and* α_i *even for all* $i \geq 2$.

- *Then* K_n *admits a 2-factorization of type* $\mathcal{T}$*.*

Signature of a 2-factorization type

- Signature of the 2-factorization type $\mathcal{T} = [T_1^{\langle\alpha_1\rangle}, \ldots, T_k^{\langle\alpha_k\rangle}]$: $[\alpha_1, \ldots, \alpha_k]$

Let $\mathcal{G}$ be an r-regular graph.

Lemma 4

Let $\mathcal{T}$ be a 2-factorization type for $\mathcal{G}$.

- *If $A = [\alpha_1, \ldots, \alpha_k]$ is a signature for $\mathcal{T}$, then $\alpha_1 + \ldots + \alpha_k = \lfloor \frac{r}{2} \rfloor$; that is, A is a refinement of $[\lfloor \frac{r}{2} \rfloor]$.*
- *Any refinement of a signature for $\mathcal{T}$ is a signature for $\mathcal{T}$.*

Lemma 5

Assume A is a refinement of A', and A' is a refinement of $[\lfloor \frac{r}{2} \rfloor]$. Assume $\mathcal{G}$ admits a 2-factorization of type $\mathcal{T}$ for every $\mathcal{T}$ with signature A that satisfies property $\mathcal{P}$.

- *Then $\mathcal{G}$ admits a 2-factorization of type $\mathcal{T}'$ for every $\mathcal{T}'$ with signature A' that satisfies property $\mathcal{P}$.*

Bipartite 2-factorizations of K_n: previous results revisited

Theorem 6 (Häggkvist; Bryant and Danziger)

Let $\mathcal{T}$ be an admissible bipartite 2-factorization type with a signature $B = [\beta_1, \ldots, \beta_s]$.
Suppose there exists a refinement A of B such that

(i) $n \equiv 2 \pmod 4$ *and* $A = [2^{\langle \frac{n-2}{4} \rangle}]$; *or*

(ii) $n \equiv 0 \pmod 4$ *and* $A = [3, 2^{\langle \frac{n-8}{4} \rangle}]$.

Then K_n admits a 2-factorization of type $\mathcal{T}$.

Bipartite 2-factorizations of λK_n: new results

Theorem 7 (Bahmanian and Šajna, 2021^+)

Let $\mathcal{T} = [T_1^{\langle \beta_1 \rangle}, \ldots, T_s^{\langle \beta_s \rangle}]$ be an admissible bipartite 2-factorization type for λK_n, with each T_i an admissible bipartite 2-factor type for K_n. Suppose there exists a refinement $[\alpha_1, \ldots, \alpha_k]$ of $[\beta_1, \ldots, \beta_s]$ such that

(i) $n \equiv 2 \pmod 4$ *and* $\left| \{i : \alpha_i \text{ is odd}\} \right| \leq \lfloor \frac{\lambda}{2} \rfloor$; *or*

(ii) $n \equiv 0 \pmod 4$, $\left| \{i : \alpha_i \text{ is odd}\} \right| \leq \lambda + \lfloor \frac{\lambda}{2} \rfloor$, *and* $\left| \{i : \alpha_i \text{ is odd}, \alpha_i \geq 3\} \right| \geq \lambda$.

Then λK_n admits a 2-factorization of type $\mathcal{T}$.

Proof of Theorem 7

- For $\lambda = 1$, apply Theorem 6.
- Hence assume $\lambda \geq 2$.
- By Lemma 4, we may assume that $\mathcal{T} = [T_1^{\langle \alpha_1 \rangle}, \ldots, T_k^{\langle \alpha_k \rangle}]$ and $[\alpha_1, \ldots, \alpha_k]$ satisfies (i) or (ii).
- Note that $\sum_{i=1}^{k} \alpha_i = \lfloor \frac{\lambda(n-1)}{2} \rfloor = \lambda \frac{n-2}{2} + \lfloor \frac{\lambda}{2} \rfloor$.

Proof of Theorem 7 cont'd

Case $n \equiv 2 \pmod 4$. Hence $\left|\{i : \alpha_i \text{ is odd}\}\right| \le \lfloor \frac{\lambda}{2} \rfloor$. Let $t = \frac{n-2}{4}$.

- As $\sum_{i=1}^{k} \alpha_i = 2\lambda t + \lfloor \frac{\lambda}{2} \rfloor$, we may assume

$$[\alpha_1, \ldots, \alpha_k] = [\underbrace{\underbrace{2, \ldots, 2}_{t}, \ldots, \underbrace{2, \ldots, 2}_{t}}_{\lambda t}, \alpha_{\lambda t+1}, \ldots, \alpha_k]$$

- By Theorem 6, K_n admits a 2-factorization of type $\mathcal{T}_j = [T_{(j-1)t+1}^{\langle 2 \rangle}, \ldots, T_{jt}^{\langle 2 \rangle}]$ for $j = 1, \ldots, \lambda$.
- Let $T_1', \ldots, T_{\lfloor \frac{\lambda}{2} \rfloor}'$ be 2-factor types such that

$$\begin{aligned} \mathcal{T} &= [T_1^{\langle \alpha_1 \rangle}, \ldots, T_k^{\langle \alpha_k \rangle}] = [T_1^{\langle 2 \rangle}, \ldots, T_{\lambda t}^{\langle 2 \rangle}] \sqcup [T_1', \ldots, T_{\lfloor \frac{\lambda}{2} \rfloor}'] \\ &= \left(\sqcup_{j=1}^{\lambda} \mathcal{T}_j \right) \sqcup [T_1', \ldots, T_{\lfloor \frac{\lambda}{2} \rfloor}']. \end{aligned}$$

- Obtain a 2-factorization of λK_n of type $\mathcal{T}$ using Theorem 1 with $\ell = \lambda$, $\mu_i = 1$ for all $i = 1, \ldots, \lambda$, and $\beta = \lambda$.

Proof of Theorem 7 cont'd

CASE $n \equiv 0 \pmod 4$. Hence $\left|\{i : \alpha_i \text{ is odd}\}\right| \le \lambda + \lfloor \frac{\lambda}{2} \rfloor$ and $\left|\{i : \alpha_i \text{ is odd}, \alpha_i \ge 3\}\right| \ge \lambda$. Let $t = \frac{n-4}{4}$.

- Note $n \ge 8$.
- As $\sum_{i=1}^{k} \alpha_i = 3\lambda + 2\lambda(t-1) + \lfloor \frac{\lambda}{2} \rfloor$, we may assume

$$[\alpha_1, \dots, \alpha_k] = [\underbrace{3, \underbrace{2, \dots, 2}_{t-1}, \dots, 3, \underbrace{2, \dots, 2}_{t-1}}_{\lambda t}, \alpha_{\lambda t+1}, \dots, \alpha_k]$$

- By Theorem 6, K_n admits a 2-factorization of type $\mathcal{T}_j = [T^{\langle 3 \rangle}_{(j-1)t+1}, T^{\langle 2 \rangle}_{(j-1)t+2}, \dots, T^{\langle 2 \rangle}_{jt}]$ for $j = 1, \dots, \lambda$.
- Let $T'_1, \dots, T'_{\lfloor \frac{\lambda}{2} \rfloor}$ be 2-factor types such that

$$\mathcal{T} = [T_1^{\langle \alpha_1 \rangle}, \dots, T_k^{\langle \alpha_k \rangle}] = \left(\sqcup_{j=1}^{\lambda} \mathcal{T}_j \right) \sqcup [T'_1, \dots, T'_{\lfloor \frac{\lambda}{2} \rfloor}].$$

- Complete using Theorem 1 as before. □

The bipartite Oberwolfach Problem for λK_n

- $OP(\mathcal{G};T)$: does $\mathcal{G}$ admit a 2-factorization of type $[T^{\langle\alpha\rangle}]$, for a 2-factor type T?

Corollary 8 (Häggkvist; Bryant and Danziger)

$OP(K_n;T)$ *has a solution for every even* n *and admissible bipartite 2-factor type* T.

Corollary 9 (Bahmanian and Šajna, 2021^+)

$OP(\lambda K_n;T)$ *has a solution for every even* n *and bipartite 2-factor type* T *that is admissible for* K_n.

The bipartite Hamilton-Waterloo Problem for λK_n

- $HWP(\mathcal{G}; T_1, T_2; \alpha_1, \alpha_2)$: does $\mathcal{G}$ admit a 2-factorization of type $[T_1^{\langle \alpha_1 \rangle}, T_2^{\langle \alpha_2 \rangle}]$, for 2-factor types T_1 and T_2?

Corollary 10 (Häggkvist; Bryant and Danziger)

Let T_1, T_2 be any distinct admissible bipartite 2-factor types for K_n. Then $HWP(K_n; T_1, T_2; \alpha_1, \alpha_2)$ has a solution for all $\alpha_1, \alpha_2 \in \mathbb{Z}^+$ such that

- $\alpha_1 + \alpha_2 = \frac{n-2}{2}$; *and*
- *at most one of α_1, α_2 is odd and $1 \notin \{\alpha_1, \alpha_2\}$.*

Theorem 11 (Bahmanian and Šajna, 2021^+)

Let T_1, T_2 be any distinct admissible bipartite 2-factor types for K_n. Then $HWP(\lambda K_n; T_1, T_2; \alpha_1, \alpha_2)$ has a solution for all $\alpha_1, \alpha_2 \in \mathbb{Z}^+$ such that

- $\alpha_1 + \alpha_2 = \lfloor \frac{\lambda(n-1)}{2} \rfloor$; *and*
- *if $\lambda = 1$, then at most one of α_1, α_2 is odd and $1 \notin \{\alpha_1, \alpha_2\}$.*

Thank you!