Spectral geometry of quantum graphs via surgery principles

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Assumptions and setup

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ compact connected metric graph: vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ finite, each edge $e \in \mathcal{E}$ corresponds to an interval of finite length
- Consider self-adjoint differential operators on $L^2(\mathcal{G})$, prototypically the *standard Laplacian*:
  - $-\Delta u \equiv -u''$ on each edge
  - vertex conditions on the functions u in the domain: u continuous on $\mathcal{G}$ and satisfies Kirchhoff condition (sum of inward-pointing derivatives at each $v \in \mathcal{V}$ is zero)
  - Could consider other conditions (Dirichlet, δ-couplings, ...)
- Sequence of eigenvalues $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots$, admit variational characterisation: e.g.

$$\mu_1 = \inf_{0 \neq u \in H^1(\mathcal{G})} \frac{\int_{\mathcal{G}} |u'|^2\,dx}{\int_{\mathcal{G}} u^2\,dx}, \qquad \mu_2 = \inf_{\substack{0 \neq u \in H^1(\mathcal{G}) \\ u \perp 1}} \frac{\int_{\mathcal{G}} |u'|^2\,dx}{\int_{\mathcal{G}} u^2\,dx}$$

Spectral geometry

How do the eigenvalues $\mu_k = \mu_k(\mathcal{G})$ depend on geometric (and topological, metric, ...) features of the graph, e.g. its total length L, its diameter D, its number of edges or vertices, its average edge length, ...

Prototypical case: $\mu_2(\mathcal{G})$, the *spectral gap*, proxy for "rate of diffusion" in graph: large $\mu_2(\mathcal{G})$ implies rapid convergence to equilibrium uniform heat distribution, small $\mu_2(\mathcal{G})$ implies slow convergence. Prototypical result:

Theorem (Nicaise, 1987; Friedlander, 2005, ...)

Let $\mathcal{G}$ have total length $L > 0$, then

$$\mu_2(\mathcal{G}) \geq \frac{\pi^2}{L^2}$$

with equality iff $\mathcal{G}$ is a path of length L (an interval modulo vertices of degree 2).

How?

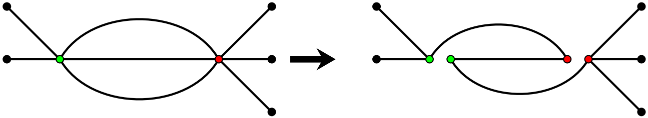
- "Doubling trick" using *Eulerian cycles* (Nicaise, 1987; Kurasov and Naboko, 2013)
- Symmetrisation argument (Friedlander, 2005), also works for the higher eigenvalues and can be refined to show $\mu_2(\mathcal{G}) \geq \frac{4\pi^2}{L^2}$ (spectral gap of a loop or equivalent graph) if $\mathcal{G}$ is *doubly connected* (Band and Lévy, 2017)
- *Surgery*: how do "local" changes to a graph affect the eigenvalues?

Examples

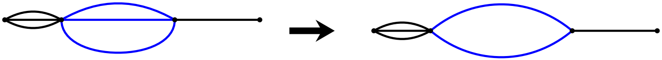
- Cutting through a vertex $(\mu_k \downarrow)$, or gluing vertices together
- Lengthening $(\mu_k \downarrow)$ or shortening an edge
- Attaching $(\mu_k \downarrow)$ or removing a pendant
- Transplantation (shifting mass from one area to another)
- Unfolding; "local" symmetrisation (of a small part of the graph)

Example: unfolding/local symmetrisation

- Replacing an *odd* number of parallel edges with one long edge always lowers μ_k:

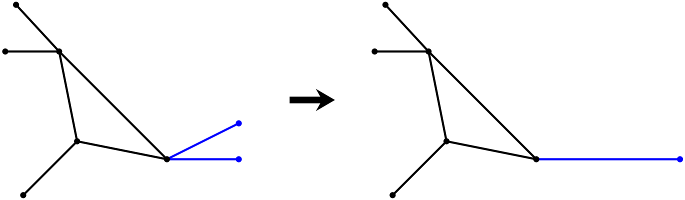


- Replacing k parallel edges with $m \leq k$ equal parallel edges lowers μ_2 *if* there is an eigenfunction monotonically increasing along each edge.

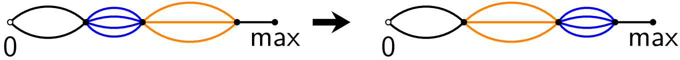


- Without the monotonicity assumption this need not hold, in particular in general it doesn't for the higher eigenvalues. Example: "equilateral 3-pumpkin" of total length 3 has eigenvalues 0, π^2, π^2, π^2, ...; a loop of length 3 has eigenvalues 0, $\frac{4\pi^2}{9}$, $\frac{4\pi^2}{9}$, $\frac{16\pi^2}{9}$, ...

- *Unfolding pendant edges:* Replacing two pendant edges at a vertex by one of the same total length always lowers μ_2



- *Transplantation:* roughly speaking, moving mass from where the eigenfunction is close to zero, to where it is big, decreases μ_2 (or first Dirichlet eigenvalue).



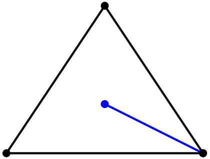
Dirichlet condition at 0 $\implies \exists$ eigenfunction monotonically increasing from left to right. This transplantation lowers the first Dirichlet eigenvalue.

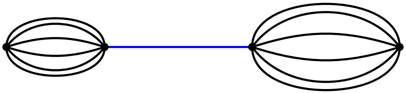
Why perform surgery?

Allows finer control, better and more comparisons: taylor to other quantities (e.g. diameter) or obtain better estimates:

Definition

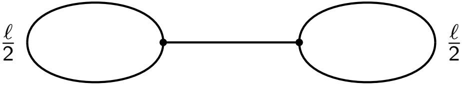
The *doubly connected part* of a graph is the largest subgraph whose every connected component is doubly connected.





Theorem (Berkolaiko–K.–Kurasov–Mugnolo, 2019)

Let $\mathcal{G}$ have total length $L > 0$ and let its doubly connected part have total length $\ell \in [0, L]$. Then $\mu_2(\mathcal{G})$ is bounded from below by the spectral gap of the *dumbbell* with total length L, whose loops have length $\ell/2$ each.



- Interpolates smoothly between the bounds of Nicaise ($\ell = 0$, minimiser is a path, $\frac{\pi^2}{L^2}$) and Band–Lévy ($\ell = L$, minimiser figure-8 equivalent to a loop, $\frac{4\pi^2}{L^2}$).
- Dumbbell eigenvalue is an increasing function of ℓ (surgery!).
- Proof of theorem combines surgery techniques:
  1. glue based on the eigenfunction to create a "pumpkin chain";
  2. locally symmetrise to make the pumpkins regular;
  3. transplant to push the fatter pumpkins outwards; and
  4. cut to create a dumbbell.

Thank you for your attention!

- G. Berkolaiko, J. B. Kennedy, P. Kurasov and D. Mugnolo, *Surgery principles for the spectral analysis of quantum graphs*, Trans. Amer. Math. Soc. **372** (2019), 5153–5197.