The Obata Theorem on a 7-D QC manifold

Abdel Mohamed & Dimiter Vassilev
University of New Mexico

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Riemannian Case

Theorem 1 (A. Lichnerowicz, 1958; M. Obata, 1962). *Suppose* (M^n, g) *is a compact Riemannian manifold satisfying the lower Ricci bound*

$$Ric(X,X) \geq (n-1)g(X,X).$$

i) *(Lichnerowicz) If* M *admits a function* f *such that* $\Delta f = \lambda f$ *and* $\lambda \neq 0$*, then* $\lambda \geq n$*.*
ii) *(Obata) If* M *admits a function* f *such that* $\Delta f = nf$ *then* M *is isometric to the round unit sphere.*

Remarks on Proofs

Decompose the Hessian of f into a trace-free part plus a multiple of g

$$\nabla^2 f = (\nabla^2 f)_{[0]} + \frac{1}{n}\langle \nabla^2 f, g\rangle\, g.$$

Integrate Bochner's formula

$$-\frac{1}{2}\Delta(|\nabla f|^2) = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + \mathrm{Ric}(\nabla f, \nabla f)$$

and use $\Delta f = \lambda f$ and the lower Ricci bound $Ric \geq (n-1)$,

$$0 \geq \int_M |(\nabla^2 f)_{[0]}|^2\, Vol_g + \frac{n-1}{n}(n-\lambda)\int_M |\nabla f|^2\, Vol_g.$$

This can only hold for a non-constant f if $\lambda \geq n$.

If $\lambda = n$, then the trace-free part of the Hessian of this f must vanish and therefore

$$\nabla^2 f = \cancelto{0}{(\nabla^2 f)_{[0]}} + \frac{1}{n}\langle \nabla^2 f, g\rangle\, g = -\frac{1}{n}(\Delta f)g.$$

Thus, if a *compact* manifold M satisfies $Ric \geq (n-1)g$, then

$$\Delta f = nf \quad \Longrightarrow \quad \nabla^2 f = -fg,$$

which allows the use of the following result characterizing the sphere using a Hessian equation for an eigenfunction.

Theorem 2 (M. Obata, 1962). *A complete Reimannian manifold admits a non-constant f such that*

$$\nabla^2 f = -fg.$$

if, and only if, it is isometric to the round unit sphere.

Bottom of spectrum in Kähler, QK and CR cases

- Lichnerowicz '58, if M is a closed Kähler manifold with $\mathrm{Ric} \geq k$, then $\lambda_1 \geq 2k$. Furthermore, equality implies that the gradient field of any eigenfunction for λ_1 is a (non-trivial) real holomorphic vector field.
- Alekseevsky and Marchiafava '95, if M^{4m} is QK and $m \geq 2$ with $Ric = 4(m+2)k > 0$, then $\lambda_1 \geq 8(m+1)k$. Moreover, the equality characterize the quaternionic projective space.
- A. Greenleaf '85 for $n \geq 3$; S.-Y. Li, & H.-S. Luk '04 for $n = 2$; H.-L. Chiu '06 for $n = 1$, if M is a compact strictly pseudoconvex pseudohermitian manifold of dimension $2n+1$ such that $Ric(X,X) + 4A(X,JX) \geq kg(X,X),\ X \in H$, then any eigenvalue λ of the *sub-Laplacian* satisfies $\lambda \geq \frac{n}{n+1}k$, provided that the CR-Paneitz operator is non-negative when $n = 1$. Equality characterizes the Sasakian sphere [S.-Y. Li, X. Wang, '13 for $n \geq 2$, w/ S. Ivanov '14 for $n = 1$].

As far as the characterization of the Sasakian sphere through a ”Hessian equation” is concerned, we have

Theorem 3 (w/ Ivanov ’12). *Let (M, θ) be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1$ with a divergence-free pseudohermitian torsion, $\nabla^* A = 0$. Assume, further, that M is complete with respect to the associated Riemannian metric. If there is a smooth function $f \not\equiv 0$ whose Hessian with respect to the Tanaka-Webster connection satisfies*

$$\nabla^2 f(X,Y) = -fg(X,Y) - df(\xi)\omega(X,Y), \qquad X, Y \in H = Ker\,\theta,$$

then up to a scaling of θ by a positive constant (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$ provided either $n \geq 2$, or $n = 1$ and the pseudohermitian torsion A vanishes.

- S. Chanillo, H.-L. Chiu, & P. Yang ’12 for $n = 1$ and positive CR Paneitz, and S.-Y. Li, D.N. Son, X. Wang for $n \geq 2$ showed analogue of the Lichnerowicz eigenvalue bound for the Kohn Laplacian $\Box_b$. Rigidity was shown in S.-Y. Li, D.N. Son, X. Wang for $n \geq 2$ and J. Case, & P. Yang ’21.

  N.B. 0 is an isolated eigenvalue of $\Box_b$ if and only if M is embeddable.

Quaternionic Contact Structure (M^{4n+3}, η)

i) co-dim three distribution H, locally, $H = \bigcap_{s=1}^{3} Ker\, \eta_s$, $\eta_s \in \Gamma(T^*M)$.

ii) H carries a quaternion structure: a 2-sphere bundle of "almost complex structures" (locally) generated by $I_s : H \to H$, $I_s^2 = -1$, satisfying $I_1 I_2 = -I_2 I_1 = I_3$;

iii) a "horizontal metric" g on H, such that for all $X, Y \in H$

$$g(I_s X, I_s Y) = g(X, Y) \qquad 2\omega_s(X, Y) \stackrel{def}{=} 2g(I_s X, Y) = d\eta_s(X, Y).$$

Reeb vector fields: $TM = H \oplus V$, for $V = span\{\xi_1, \xi_2, \xi_3\}$ where

$$\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$$

If $n = 1$, assume that the Reeb vector fields exist [**Duchemin, D.**].

The Biquard connection: There exists a unique linear connection ∇ on M with the properties: (i) V and H are parallel; (ii) g and $\Omega = \sum_{j=1}^{3} \omega_j \wedge \omega_j$ are parallel; (iii) the torsion satisfies

a) $\forall X, Y \in H, \quad T(X, Y) = -[X, Y]|_V = 2\omega_i(X, Y)\xi_i \in V$

b) $\forall \xi \in V,\ X \in H, T_\xi(X) \equiv T(\xi, X) \in H$ and $T_\xi \in (sp(n) + sp(1))^\perp$, $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$. $T^0_{\xi_j}$-symmetric, $I_j U$-skew-symmetric..

We extend the horizontal metric g to a Riemannian metric h on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_t) = \delta_{st}$. **N.B.** h as well as the Biquard connection do not depend on the action of $SO(3)$ on V. Throughout, we will use the following conventions.

- qc-curvature: $\mathcal{R}(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$;
- qc-Ricci tensor: $Ric(A,B) = \mathcal{R}(e_a, A, B, e_a) \overset{def}{=} \sum_{a=1}^{4n} h(\mathcal{R}(e_a, A)B, e_a)$;
- qc-scalar curvature: $\mathrm{Scal} = tr_H\, Ric = Ric(e_a, e_a)$; normalized qc-scalar curvature

$$S = \frac{\mathrm{Scal}}{4n(n+2)}.$$

We have the following properties.

- If $T^0 \overset{def}{=} T^0_{\xi_i} I_i$, then by $T^0 \in \Psi_{[-1]}$ and

$$Ric = (2n+2)T^0 + (4n+10)U + \frac{\mathrm{Scal}}{4n} g.$$

- M is called ***qc-Einstein*** if $T^0 = U = 0$. This is equivalent to M being locally 3-Sasakian. For a qc-Einstein we know that $\mathrm{Scal} = const$ [w/ Ivanov & Minchev '10 & '17]

qc Volume and divergence formula

Vol_η will denote the volume form,

$$\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \Omega^n,$$

where $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$ is the fundamental 4-form.

Note the integration by parts formula

$$\int_M (\nabla^* \sigma)\ Vol_\eta \ = \ 0, \tag{1}$$

where the (horizontal) divergence of a 1-form σ is given by
$\nabla^*\sigma = -tr|_H \nabla\sigma = -\nabla\sigma(e_a, e_a)$ for an orthonormal frame $\{e_a\}_{a=1}^{4n}$ of the horizontal space.

The sub-Laplacian and horizontal Hessian

For a smooth function f we define

- the "horizontal" gradient $\nabla f \in \Gamma(H)$ using $df(X) = g(\nabla f, X)$.
- the horizonal Hessian, $\nabla^2 f$,

$$\nabla^2 f(Y, X) = g(\nabla_Y(\nabla f), X),\ X, Y \in \Gamma(H).$$

- the (positive) sub-Laplacian is $\Delta f = -\mathrm{tr}^g(\nabla^2 f) = -\langle \nabla^2 f, g\rangle$;
- the identity $\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2\sum \omega_s(X, Y) df(\xi_s)$ gives

$$\langle \nabla^2 f, \omega_s\rangle = -4n df(\xi_s).$$

Remark The sub-Laplacian is a sub-elliptic operator, hence has a discrete spectrum on any compact manifold.

For $\Psi \in \mathrm{End}(H)$, the two $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant components are

$$\Psi_{[3]} := \Psi^{+++}, \quad \Psi_{[-1]} := \Psi^{+--} + \Psi^{+-+} + \Psi^{---},$$

where Ψ^{+++} commutes with all three I_s, Ψ^{+--} commutes with I_1 and anti-commutes with the others, etc. The same notation is used for the corresponding $(2,0)$-tensors, $\Psi_{[3]}(\cdot,\cdot) := g(\Psi_{[3]}\cdot,\cdot)$. In particular, for the Hessian of f we have the components,

$$(\nabla^2 f)_{[3]}(X,Y) = \frac{1}{4}\left[\nabla^2 f(X,Y) + \sum_{s=1}^{3} \nabla^2 f(I_s X, I_s Y)\right],$$

$$(\nabla^2 f)_{[-1]}(X,Y) = \frac{1}{4}\left[3\nabla^2 f(X,Y) - \sum_{s=1}^{3} \nabla^2 f(I_s X, I_s Y)\right].$$

Let $(\nabla^2 f)_{[\cdot][0]}$ denote the trace-free part of the corresponding component, so that,

$$(\nabla^2 f)_{[3]} = (\nabla^2 f)_{[3][0]} + \frac{1}{4n}\langle \nabla^2 f, g\rangle\, g \;\Rightarrow\; |(\nabla^2 f)_{[3]}|^2 \geq \frac{1}{4n}\langle \nabla^2 f, g\rangle^2$$

and

$$\begin{aligned}(\nabla^2 f)_{[-1]} = (\nabla^2 f)_{[-1][0]} + \frac{1}{4n}\sum_{s=1}^{3}\langle \nabla^2 f, \omega_s\rangle\, \omega_s \\ \Rightarrow \quad |(\nabla^2 f)_{[-1]}|^2 \geq \frac{1}{4n}\sum_{s=1}^{3}\langle \nabla^2 f, \omega_s\rangle^2.\end{aligned}$$

Therefore, we have the Hessian inequality

$$|\nabla^2 f|^2 \geq \frac{1}{4n}\left[\langle \nabla^2 f, g\rangle^2 + \sum_{s=1}^{3}\langle \nabla^2 f, \omega_s\rangle^2\right].$$

The qc-Paneitz type operator

Let $\{e_a\}_{a=1}^{4n}$ denote a local orthonormal frame of H.

- The $P-$form of a fixed smooth function f on M is defined by

$$P_f(X) = \sum_{a=1}^{4n} \nabla^3 f(X, e_a, e_a) + \sum_{t=1}^{3} \sum_{a=1}^{4n} \nabla^3 f(I_t X, e_a, I_t e_a) - 4nSdf(X) + 4nT^0(X, \nabla f) - \frac{8n(n-2)}{n-1} U(X, \nabla f),$$

- The $\mathcal{C}-$operator of M is the 4-th order differential operator

$$f \mapsto \mathcal{C}f = -\nabla^* P_f = \sum_{a=1}^{4n} (\nabla_{e_a} P_f)(e_a).$$

- We say that $\mathcal{C}-$operator is non-negative, $\mathcal{C} \geq 0$, if for any f we have

$$\int_M f \cdot \mathcal{C}f \, Vol_\eta = -\int_M P_f(\nabla f) \, Vol_\eta \geq 0.$$

Theorem 4 (w/ Ivanov & Petkov, '13). *Let M be compact qc manifold.*
a) $\mathcal{C} \geq 0$ *for* $n > 1$*. Furthermore* $\mathcal{C}f = 0$ *iff* $(\nabla^2 f)_{[3][0]}(X,Y) = 0$,

$$\nabla^2 f(X,Y)+\nabla^2 f(I_1X,I_1Y)+\nabla^2 f(I_2X,I_2Y)+\nabla^2 f(I_3X,I_3Y) = -\frac{1}{4n}(\triangle f)\, g(X,Y).$$

In this case the $P-$*form of* f *vanishes as well.*
b) If $n = 1$ *and M is qc-Einstein with Scal* ≥ 0*, then the* $\mathcal{C}$*-operator on (or* $P-$*function of) an eigenfunction of the sub-Laplacian is non-negative,*

$$\triangle f = \lambda f \qquad \Rightarrow \qquad \int_M f \cdot \mathcal{C}f \, Vol_\eta = -\int_M P_f(\nabla f)\, Vol_\eta \geq 0.$$

When $n = 1$ we *assume the extra condition* of the non-negativity of the $\mathcal{C}$-operator on an eigenfunction.

Proof of part a) where $n > 1$

For $n > 1$, the divergence of $(\nabla^2 f)_{[3][0]}$ is proportional to the P-form of f,

$$-\nabla^*[(\nabla^2 f)_{[3][0]}](X) = \frac{n-1}{4n} P_f(X).$$

When M is compact, it follows

$$\frac{n-1}{4n} \int_M f \cdot \mathcal{C} f \, \mathrm{Vol}_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2 \mathrm{Vol}_\eta \geq 0.$$

Hence, the $\mathcal{C}$-operator is non-negative when M is compact and $n > 1$.

The qc-Lichnerowicz Theorem

Theorem 5 (w/ Ivanov & Petkov '13, '14). *Suppose* $(M^{4n+3}, g, \mathbb{Q})$ *is a compact qc-manifold satisfying the Lichernowicz-type bound*

$$\mathcal{L}(X,X) \stackrel{def}{=} 2Sg(X,X) + \alpha_n T^0(X,X) + \beta_n U(X,X) \geq 4g(X,X).$$

When $n = 1$ *assume, in addition, the non-negativity of the* $\mathcal{C}$*-operator of any eigenfunction. Then, any eigenvalue* λ *of the sub-Laplacian* Δ *satisfies the inequality* $\lambda \geq 4n$*.*

Remark The constants α_n and β_n are the same as in the convexity of the entropy for positive solutions of the "heat" equation with $\mathcal{L} \geq 0$, [—, '17].

$$\alpha_n = \frac{2(2n+3)}{2n+1}, \quad \beta_n = \frac{4(2n-1)(n+2)}{(2n+1)(n-1)}$$

Proof of the qc-Licnerowicz Theorem, $n \geq 1$

Integrating the qc-Bochner formula

$$\frac{1}{2}\Delta(|\nabla f|)^2 = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + 2(n+2)\left(S|\nabla f|^2 + T^0(\nabla f, \nabla f)\right) + 4(n+1)U(\nabla f, \nabla f) + 4\sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f)$$

the Hessian inequality and Lichernowicz' type bound, $\mathcal{L}(\nabla f, \nabla f) \geq 4|\nabla f|^2$, give

$$0 \geq \int_M \left(|\nabla^2 f|^2 - \frac{1}{4n}\left[\langle \nabla^2 f, g\rangle^2 + \sum_{s=1} \langle \nabla^2 f, \omega_s\rangle^2 \right]\right) \mathrm{Vol}_\eta - \frac{3}{4n}\int_M P_f(\nabla f) \mathrm{Vol}_\eta + \frac{2n+1}{2n}\int_M \mathcal{L}(\nabla f, \nabla f) - \frac{\lambda}{n}|\nabla f|^2 \, \mathrm{Vol}_\eta \geq \frac{2n+1}{8n}(4n-\lambda)\int_M |\nabla f|^2 \, \mathrm{Vol}_\eta,$$

using the hypothesis on the $\mathcal{C}$-operator of an eigenfunction when $n=1$. Hence, $\lambda \geq 4n$.

First equations in the equality case of the qc-Licnerowicz estimate, $n \geq 1$

Therefore, on a compact qc-manifold with a Lichnerowicz-type bound and non-negative $\mathcal{C}$-operator on any eigenfunction when $n = 1$, any eigenfunction f such that $\Delta f = 4nf$, satisfies the system (on H)

$$\nabla^2 f = \frac{1}{4n} \langle \nabla^2 f, g \rangle \, g + \frac{1}{4n} \sum_{s=1}^{3} \langle \nabla^2 f, \omega_s \rangle \, \omega_s = -fg - \sum_{s=1}^{3} df(\xi_s) \omega_s,$$

$$\mathcal{L}(\nabla f, \nabla f) - 4|\nabla f|^2 = 2 \left[(S-2)|\nabla f|^2 + \frac{5}{3} T^0(\nabla f, \nabla f) \right] = 0,$$

$$\int_M P_f(\nabla f) \, Vol_\eta = 0.$$

The qc Obata type theorem

Theorem 6 (w/ Ivanov & Petkov '15). *Suppose* (M^{4n+3}, η) *is a compact qc-manifold satisfying the Lichnerowicz-type bound*

$$\mathcal{L}(X,X) \overset{def}{=} 2Sg(X,X) + \alpha_n T^0(X,X) + \beta_n U(X,X) \geq 4g(X,X).$$

When $n = 1$ *assume also the non-negativity of the* $\mathcal{C}$*-operator of any eigenfunction. Then,* M *admits a function* f *such that* $\Delta f = 4nf$ *only if*

i) M *is qc-homothetic to the* 3*-Sasakian sphere when* $n > 1$*;*

ii) M *is qc-homothetic to the* 3*-Sasakian sphere when* $n = 1$ *and* M ***is qc-Einstein****.*

Remark The goal here is to show that the above additional hypothesis on M being qc-Einstein when $n = 1$ was unnecessary. To show that M is qc-Einstein we must show that $T^0 = 0$ since $U = 0$ in 7-D.

The qc Obata type result for the Hessian when $n > 1$

Similar to the Riemannian case, when $n > 1$ the proof of the qc Obata type theorem relies on an analogous theorem in the qc-setting concerning complete manifolds that admit functions with Hessian as (recalled) below.
Theorem 7 (w/ Ivanov & Petkov '15). *Let* $(M^{4n+3}, \mathbb{Q}, \eta)$*,* $n > 1$*, be a qc-manifold that is complete with respect to the Riemannian metric*

$$h = g + \sum_{s=1}^{3} \eta^s \otimes \eta^s.$$

Then, M admits a non-constant f such that

$$\nabla^2 f = -fg - \sum_{s=1}^{3} df(\xi_s)\omega_s$$

if, and only if, M is qc-homothetic to the unit 3*-Sasakian sphere.*

Remark on the proof of the qc Obata type result for $n > 1$

Part 1: show $T^0 = 0$ and $U = 0$, i.e., M is qc-Einstein.

i) Find the remaining parts of the Hessian in terms of the torsion tensors.

ii) A simple argument shows that $T^0(I_s\nabla f, \nabla f) = U(I_s\nabla f, \nabla f) = 0$ and $T^0(I_s\nabla f, I_t\nabla f) = 0$, $s,\ t \in \{1, 2, 3\}$, $s \neq t$.

iii) Determine the torsion tensors T^0 and U in terms of ∇f and the tensor $U(\nabla f, \nabla f)$. For e.g.,

$$|\nabla f|^4 T^0(X, Y) = -\frac{2n}{n-1} U(\nabla f, \nabla f)\Big[3df(X)df(Y) - \sum_{s=1}^{3} df(I_sX)df(I_sY)\Big].$$

iv) Formulas of the same type for ∇T^0 and ∇U. **N.B.** In particular, $L_{\nabla f}|U|^2 = \frac{4(n-1)}{n+2} f|U|^2$ as in the Riemannian case for Ric_0!.

v) The crux is the proof that $U(\nabla f, \nabla f) = 0$ Use Ricci's identities, the contracted Bianchi second identity and many properties of the torsion of a qc-manifolds to obtain
$0 = \nabla^3 f(\xi_i, I_i\nabla f, \nabla f) - \nabla^3 f(I_i\nabla f, \nabla f, \xi_i) = \frac{2}{n+2} fU(\nabla f, \nabla f)$.

vi) When $n > 1$, the "Hessian eq'n" implies an elliptic PDE
$\triangle^h f = (4n + 3)f + \frac{n+1}{n(2n+1)}(\nabla^* T^0)(\nabla f) + \frac{3}{(2n+1)(n-1)}(\nabla^* U)(\nabla f)$.

The qc Obata type result in 7-D

Theorem 8 (w/ Abdel Mohamed '20). *Suppose (M^7, η) is a compact 7-D qc-manifold satisfying the Lichnerowicz-type bound*

$$\mathcal{L}(X,X) = 2Sg(X,X) + \frac{10}{3}T^0(X,X) \geq 4g(X,X), \qquad X \in \Gamma(H),$$

and the P-function of any eigenfunction associated to the first non-zero eigenvalue of the sub-Laplacian is non-negative. If the (lowest) eigenvalue of the sub-Laplacian is 4, then (M,η) is qc-Einstein.

Remark *The result concerning qc-manifolds that are complete with respect to the extended Riemannian metrics and support a function satisfying the horizontal Hessian equation is open in dimension 7.*

Since T^0 is symmetric, and H is 4-dimensional, it has $4+3+2+1=10$ components that we need to show vanish. When $n=1$ we have trivially $T^0 = T^0_{[-1]}$ since $T^0_{[3]} = 0$. In addition we must determine S. Overall, we have 10 ”unknowns”.

Equation for $T^0(X, \nabla f)$

Lemma 9. *Define the quadratic symmetric (0,2)-tensor* $\mathcal{P}$ *by*

$$\mathcal{P}(X,Y) \stackrel{def}{=} 2\left[\mathcal{L}(X,Y) - 4g(X,Y)\right] = 4\left[(S-2)g(X,Y) + \frac{5}{3}T^0(X,Y)\right].$$

The P-form of f is $\mathcal{P}(X, \nabla f)$, *i.e.,*

$$P_f(X) = \mathcal{P}(X, \nabla f) = 4(S-2)df(X) + \frac{20}{3}T^0(X, \nabla f).$$

Furthermore, $\mathcal{P}(X, \nabla f) = 0$, *hence* $T^0(X, \nabla f) = -\frac{3}{5}(S-2)df(X)$.

The Lichnerowicz-type bound implies that $\mathcal{P}$ is non-negative $\mathcal{P}(X,X) \geq 0$, hence, taking into account that T^0 is a traceless tensor, we have $S \geq 2$, while since we are in the case of equality we can obtain a point-wise identity $\mathcal{P}(\nabla f, \nabla f) = 0$.

The $\{I_\alpha \nabla f\}$ frame

When $n = 1$ we can frame the 4-dimensional horizontal space with

$$\{I_\alpha \nabla f\}_{\alpha=0}^{3} = \{I_0 \nabla f, I_1 \nabla f, I_2 \nabla f, I_3 \nabla f\}, \quad I_0 = \mathrm{id}_H$$

at points where $|\nabla f| \neq 0$. We *will show* that the horizontal gradient cannot vanish on any open set. Using this frame the problem becomes showing that for $\alpha, \beta \in \{0, 1, 2, 3\}$, we have

$$T_{\alpha\beta} := T^0(I_\alpha \nabla f, I_\beta \nabla f) = 0.$$

So far we have:

- from $P_f(X) = T^0(X, \nabla f) + \frac{3}{5}(S-2)df(X) = 0$ it follows that $T_{0i} = T_{i0} = 0$, $i = 1, 2, 3$;
- from $T^0_{[3]} = 0$, we have $T_{00} + T_{11} + T_{22} + T_{33} = 0$.

Hence, we are down to determining 7 functions, coming from T^0 and S.

We'll denote the derivative of $T^0(X,Y)$ along $I_\gamma \nabla f$, evaluted on the $\{I_\alpha \nabla f\}_{\alpha=0}^3$ frame for H, by

$$T_{\alpha\beta;\gamma} := \nabla T^0(I_\gamma \nabla f, I_\alpha \nabla f, I_\beta \nabla f)$$

and the derivative of f along the $\{\xi_s\}_{s=1}^3$ frame for V by

$$f_s := df(\xi_s).$$

- The symmetry of T^0 and the fact that the Biquard connection preserves the type of tensor, imply that $T_{\alpha\beta;\gamma} = T_{\beta\alpha;\gamma}$.
- A computation shows that

$$T_{i0;0} = -f_i T_{00} + f_i T_{ii} + f_j T_{ij} + f_k T_{ik} = 0, \quad f_i T_{00} = f_1 T_{i1} + f_2 T_{i2} + f_3 T_{i3}.$$

- The covariant derivative of $P_f(X) = 0$ and the Hessian equation yield

$$\nabla T^0(Y,X,\nabla f) = -\frac{3}{5} dS(Y) df(X) + f\left(T^0(Y,X) + \frac{3}{5}(S-2) g(Y,X)\right) + \sum_{s=1}^{3} f_s \left(T^0(I_s Y, X) + \frac{3}{5}(S-2)\omega_s(Y,X)\right).$$

Why we need another relation for $\nabla T^0(Y, I_j\nabla f, I_i\nabla f)$?

Since the canonical connection preserves the type of a tensor and T^0 is symmetric, we can compute terms of the form $\nabla T^0(Y, \nabla f, I_i\nabla f)$ by finding $\nabla T^0(Y, I_i\nabla f, \nabla f)$ from the above formula, i.e., from

$$\nabla T^0(Y,X,\nabla f) = -\frac{3}{5}dS(Y)df(X) + f\left(T^0(Y,X) + \frac{3}{5}(S-2)g(Y,X)\right) + \sum_{s=1}^{3} f_s\left(T^0(I_sY,X) + \frac{3}{5}(S-2)\omega_s(Y,X)\right).$$

- We cannot obtain in this way formulas for the $\nabla T^0(Y, I_j\nabla f, I_i\nabla f)$.
- We also still need to show that $I_\alpha\nabla f$ can be used as a frame globally.

A key relation for ∇T^0 – the "5-3" formula

Lemma 10. *For any cyclic permutation of (i,j,k) define the tensors*

$$\Gamma^i(Y,X) \stackrel{def}{=} \omega_j(Y,X)\rho_k(I_i\nabla f,\xi_i) - \omega_k(Y,X)\rho_j(I_i\nabla f,\xi_i)$$
$$+df(I_kY)\rho_j(I_iX,\xi_i)+df(I_kX)\rho_j(I_iY,\xi_i)-df(I_jY)\rho_k(I_iX,\xi_i)-df(I_jX)\rho_k(I_iY,\xi_i),$$

using the Ricci 2-forms $\rho_i(A,B)=\frac{1}{4n}R(A,B,e_a,I_ie_a)$. The following identity holds true,

$$5\nabla T^0(Y,X,I_i\nabla f)-3\nabla T^0(X,Y,I_i\nabla f) = -3[\nabla T^0(\nabla f,Y,I_iX)+\nabla T^0(\nabla f,I_iY,X)]$$
$$+3dS(Y)df(I_iX)-\frac{9}{5}dS(X)df(I_iY)+\frac{6}{5}(4+3S)f_ig(Y,X)+12\sum_{s=1}^{3}\nabla^2 f(\xi_i,\xi_s)\omega_s(Y,X)$$
$$-\frac{12}{5}(1+2S)\left[f\omega_i(X,Y)+\sum_{s=1}^{3}f_s\omega_s(Y,I_iX)\right]+f[5T^0(X,I_iY)-3T^0(I_iX,Y)]$$
$$+f_i[6T^0(I_iX,I_iY)-8T^0(X,Y)]+f_j[5T^0(X,I_kY)+6T^0(I_jX,I_iY)+3T^0(I_kX,Y)]$$
$$+f_k[6T^0(I_kX,I_iY)-5T^0(X,I_jY)-3T^0(I_jX,Y)]-12\Gamma^i(Y,X).$$

Some corollaries from the "5-3" formula for ∇T^0

In short, we arrive at an equation of the form

$$5\nabla T^0(Y,X,I_i\nabla f) - 3\nabla T^0(X,Y,I_i\nabla f) = A_i(Y,X).$$

- ***Lemma 11.*** *We have $|\nabla f| \neq 0$ almost everywhere.*
  Indeed, using the horizontal Hessian equation, the relation between the Biquard and Riemannian curvatures, the key formula for ∇T^0 we can show the following elliptic PDE,

  $$\Delta^h f = \left(\frac{23+6S}{5}\right) f - \frac{1}{6}\sum_{s=1}^{3}\left(\nabla_s^* T^0(I_s\nabla f) + \frac{3}{8} dS(I_s\nabla f)\right).$$

  Aronszajn's unique continuation implies that f cannot vanish on any open set, otherwise $f \equiv 0$, contradicting $\Delta f = 4f$.
- We can compute the same derivative, say $T_{\alpha i,\gamma}$, in two different ways, which allows to find further non-trivial relations, for example,

  $$T_{ij;0} = \frac{1}{4}[f_k(T_{ii} - T_{jj}) + f_j T_{kj} - f_i T_{ki}],$$

  $$dS(\nabla f) = 0, \qquad T_{00;0} = 0, \quad T_{ii;0} = \frac{1}{2}[f_j T_{ki} - f_k T_{ji}].$$

The normalized scalar curvature $S = 2$

- ***Lemma 12.****a) The next identities hold at almost every point of M for any cyclic permutation of (i, j, k),*

$$dS(I_i \nabla f) = -2(S-2)f_i, \qquad dS(\xi_i) = 0.$$

b) The normalized qc-scalar curvature S is constant, in fact, $S = 2$. In particular,

$$T_{00} = 0, \qquad T_{11} + T_{22} + T_{33} = 0, \quad f_1 T_{i1} + f_2 T_{i2} + f_3 T_{i3} = 0,$$

$$f_k T_{ik} - f_i T_{kk} = f_i T_{jj} - f_j T_{ij}.$$

Remark For the part $S = 2$, use $dS|_H$ to obtain a formula for $\nabla^2 S$. Then, the Ricci identity gives

$$0 = -4dS(\xi_i) = \langle \nabla^2 S, \omega_i \rangle = -2f dS(I_i \nabla f)$$

but since $f \neq 0$ a.e. we must have $dS|_H = 0$. Hence $S = \text{const}$. An argument involving
$0 = Xf_i = \nabla^2 f(X, \xi_i) - \alpha_j(X) f_k + \alpha_k(X) f_j = \nabla^2 f(X, \xi_i)$ and $S \geq 2$ shows that $S = 2$.

Taking into account $S=2$ and the "5-3" formula written as

$$16\nabla T^0(Y,X,I_i\nabla f)=5A_i(Y,X)+3A_i(X,Y)$$

we obtain the components of the divergences $\nabla^*_i T^0(X)=\nabla T^0(e_\gamma,I_ie_\gamma,X)$

Lemma 13. *The divergences of the torsion satisfy the following identities,*

$$|\nabla f|^2\nabla^*_i T^0(I_i\nabla f)=-4[fT_{ii}+f_kT_{ij}-f_jT_{ki}],$$

$$|\nabla f|^2\nabla^*_j T^0(I_k\nabla f)=|\nabla f|^2\nabla^*_k T^0(I_j\nabla f)=-4[fT_{jk}+f_jT_{ij}-f_iT_{jj}].$$

Lemma 14. *The next identities hold at almost every point,*

$$fT_{jk}=\frac{1}{4}[f_iT_{kk}-f_kT_{ki}]=\frac{1}{4}[f_jT_{ij}-f_iT_{jj}],\quad fT_{ii}=\frac{1}{4}[f_kT_{ij}-f_jT_{ki}].$$

Vanishing of the torsion

We have $T_{00} = T_{0i} = 0$, hence

$$|\nabla f|^4 |T^0|^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}^2 + 2T_{23}^2 + 2T_{31}^2 = \sum_{(ijk)} [T_{ii}^2 + 2T_{ij}^2] = \sum_{(ijk)} [T_{ii}^2 + 2T_{jk}^2],$$

where $\sum_{(ijk)}$ indicates a cyclic sum. Using the identities
$4fT_{jk} = f_i T_{kk} - f_k T_{ki} = f_j T_{ij} - f_i T_{jj}$ and $4fT_{ii} = f_k T_{ij} - f_j T_{ki}$,we obtain

$$\begin{aligned} 4f|\nabla f|^4 |T^0|^2 &= \sum_{(ijk)} \left[T_{ii}\left(f_k T_{ij} - f_j T_{ki}\right) + T_{jk}\left(f_i T_{kk} - f_k T_{ki}\right) + T_{jk}\left(f_j T_{ij} - f_i T_{jj}\right)\right] \\ &= \sum_{(ijk)} \left[f_k T_{ii} T_{ij} - f_j T_{ii} T_{ki} + f_i T_{jk} T_{kk} - f_k T_{jk} T_{ki} + f_j T_{jk} T_{ij} - f_i T_{jk} T_{jj}\right] \\ &= \sum_{(ijk)} \left[f_k T_{ii} T_{ij} - f_k T_{jj} T_{ij} + f_k T_{ij} T_{jj} - f_k T_{jk} T_{ki} + f_k T_{ki} T_{jk} - f_k T_{ij} T_{ii}\right] = 0. \end{aligned}$$

Thus, M is a qc-Einstein structure.

An application of the qc-Obata theorem

Definition 15. *A vector field Q on a qc manifold (M, η) is a qc vector field if its flow preserves the horizontal distribution $H = \ker \eta$.*

- This is equivalent to preserving the qc structure $\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta$, where ν is a smooth function and $O \in so(3)$ is a matrix valued function with smooth entries, [w/ S. Ivanov & I. Minchev '14].
- For any qc vector field Q on M we have, [w/ S. Ivanov & I. Minchev arXiv:1504.03142],

$$\Delta(\nabla^* Q_H) \;=\; -\frac{n}{2(n+2)} Q(\mathrm{Scal}) \;-\; \frac{\mathrm{Scal}}{4(n+2)} \nabla^* Q_H,$$

Theorem 16 (w/ S. Ivanov & I. Minchev arXiv:1504.03142). *Let $(M, \bar\eta)$ be a compact locally 3-Sasakian qc manifold of qc-scalar curvature $16n(n+2)$. If $\eta = 2h\bar\eta$ is qc-conformal to $\bar\eta$ structure which is also qc-Einstein, then the function h is constant unless $(M, \bar\eta)$ is the unit 3-Sasakian sphere.*

N.B. Much stronger result is proven in arXiv:1504.03142.

Idea of proof

- The following is a qc vector field,

$$Q = \frac{1}{2}\nabla f + \sum_{s=1}^{3} dh(\xi_s)\xi_s, \qquad f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2.$$

- The function $\phi = \frac{1}{2}\triangle f$ is either (i) an eigenfunction of the sub-Laplacian with eigenvalue $-4n$, $\triangle\phi = -4n\phi$, or (ii) $\phi \equiv 0$.
  (i) Use the qc Obata to see the 3-Sasakian sphere
  (ii) $\triangle f = 0$, hence by compactness and th qc Yamabe equation, $h = 1/2$.

Thank You!

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