Asymptotic behavior of dispersive electromagnetic waves in bounded domains

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Dispersive electromagnetic waves

Let Ω be in a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ. The Maxwell equations in Ω are given by

$$\begin{cases} D_t - \text{curl } H = 0 & \text{in} \quad Q := \Omega \times (0, +\infty), \\ B_t + \text{curl } E = 0 & \text{in} \quad Q, \end{cases} \tag{\textbf{P}}$$

where E and H are respectively the electric and magnetic fields, while D and B are respectively the electric and magnetic flux densities.

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where E and H are respectively the electric and magnetic fields, while D and B are respectively the electric and magnetic flux densities.
In case of electric and magnetization effects, these latter ones take the form

$$\begin{aligned} D(x,t) &= \varepsilon(x)E(x,t) + P(x,t), \\ B(x,t) &= \mu(x)H(x,t) + M(x,t), \end{aligned}$$

where ε (resp. μ) is the permittivity (resp. permeability) of the medium, while P (resp. M) is the retarded electric polarization (resp. magnetization).

The retarded electric polarization and magnetization, P and M, in most applications (see [Cassier, Joly & Kachanovska (2017), Kristensson, Karlsson & Rikte (2002), Sihvola (1999)]) are of integral form

$$P(x,t) = \int_0^t \nu_E(t-s,x)E(x,s)\,ds,$$
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where $\nu_E(t,x)$ (resp. $\nu_H(t,x)$) is the electric (resp. magnetic) susceptibility kernel.

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Our goal is to analyze the general system **(P)**, supplemented with the electric boundary conditions

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and initial conditions

$$E(\cdot,0) = E_0,\ H(\cdot,0) = H_0 \quad \text{in } \Omega, \qquad (\mathbf{IC})$$

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In this paper we restrict to the following case:

- ϵ and μ positive constants;
- $\nu_E(t,x) = \nu_E(t)$ and $\nu_H(t,x) = \nu_H(t)$.

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- ϵ and μ positive constants;
- $\nu_E(t,x) = \nu_E(t)$ and $\nu_H(t,x) = \nu_H(t)$.

This already corresponds to a large class of physical examples, see e.g. [Kristensson, Karlsson & Rikte (2002), Sihvola (1999)].

Dispersive electromagnetic wave

We further assume that

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Again these assumptions cover a large class of physical models.

For brevity, we define the function w by

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Well-posedness

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Assuming for the moment that the solution (E, H) of this system with boundary conditions **(BC)** and initial conditions **(IC)** exists, then for all $(t, s) \in [0, \infty) \times (0, \infty)$ we introduce the cumulative past histories

$$\eta_E^t(\cdot, s) = \int_0^{\min\{s,t\}} E(\cdot, t-y)\,dy,$$
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$$\lim_{s \to 0} \eta_E^t(\cdot, s) = \lim_{s \to 0} \eta_H^t(\cdot, s) = 0,$$

and the initial condition

$$\eta_E^0(\cdot, s) = \eta_H^0(\cdot, s) = 0.$$

Since formal integration by parts yields the identities

$$\int_0^t \nu_E'(t-s)E(\cdot,s)\,ds = -\int_0^\infty \nu_E''(s)\eta_E^t(\cdot,s)\,ds,$$
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the above integro-differential Maxwell system is (formally) equivalent to

$$\begin{cases} \varepsilon E_t + \nu_E(0)E - \int_0^\infty \nu_E''(s)\eta_E^t(\cdot,s)\,ds - \operatorname{curl} H = 0 \quad \text{in } Q, \\ \mu H_t + \nu_H(0)H - \int_0^\infty \nu_H''(s)\eta_H^t((\cdot,s)\,ds + \operatorname{curl} E = 0 \quad \text{in } Q, \end{cases}$$

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we obtain the abstract Cauchy problem

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \qquad \textbf{(PA)}$$

where

$$\mathcal{A}\begin{pmatrix} E \\ H \\ \eta_E \\ \eta_H \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}(-\nu_E(0)E + \int_0^\infty \nu_E''(s)\eta_E(\cdot,s)\,ds + \mathrm{curl}\,H) \\ \mu^{-1}(-\nu_H(0)H + \int_0^\infty \nu_H''(s)\eta_H(\cdot,s)\,ds - \mathrm{curl}\,E) \\ -\partial_s\eta_E(\cdot,s) + E \\ -\partial_s\eta_H(\cdot,s) + H \end{pmatrix},$$

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The existence of a solution to **(PA)** is obtained by using semigroup theory in the appropriate Hilbert setting described here below.

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The existence of a solution to **(PA)** is obtained by using semigroup theory in the appropriate Hilbert setting described here below. First we introduce the Hilbert spaces

$$J(\Omega) = \{\chi \in L^2(\Omega)^3 |\ \operatorname{div}\chi = 0\},$$

and

$$\hat{J}(\Omega) = \{\chi \in J(\Omega) |\ \chi \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

recalling that for a field $\chi \in J(\Omega)$, $\chi \cdot \mathbf{n}$ has a meaning as an element of $H^{-\frac{1}{2}}(\Gamma)$, see [Girault & Raviart (1986)].

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Observe that $w \in L^\infty([0,\infty))$ and recall that for a Hilbert space X with inner product $(\cdot,\cdot)_X$ and induced norm $\|\cdot\|_X$, $L^2_w((0,\infty);X)$ is the Hilbert space comprised of functions η defined on $(0,\infty)$ with values in X such that

$$\int_0^\infty \|\eta(s)\|_X^2 w(s)\, ds < \infty,$$

with the natural inner product

$$\int_0^\infty (\eta(s),\eta'(s))_X w(s)\, ds, \ \forall \eta,\eta' \in L^2_w((0,\infty);X).$$

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Let us notice that $L^2_w((0,\infty);X)$ is quite large since it contains all polynomials with coefficients in X,

namely for any non-negative integer n and any $a_i \in X$, $i = 0, \cdots, n$, the polynomial p defined by

$$p(s) = \sum_{i=0}^{n} a_i s^i, \quad \forall\, s \in (0, \infty),$$

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$$\mathcal{H} = J(\Omega) \times \hat{J}(\Omega) \times L^2_w((0,\infty); J(\Omega)) \times L^2_w((0,\infty); \hat{J}(\Omega)),$$

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$$((E, H, \eta_E, \eta_H)^\top, (E', H', \eta'_E, \eta'_H)^\top)_{\mathcal{H}} := \int_\Omega (\varepsilon E \cdot \bar{E}' + \mu H \cdot \bar{H}')\, dx$$
$$+ \int_0^\infty \int_\Omega (\eta_E(x,s) \cdot \bar{\eta}'_E(x,s) + \eta_H(x,s) \cdot \bar{\eta}'_H(x,s))\, dx\, w(s)\, ds,$$

for all $(E, H, \eta_E, \eta_H)^\top, (E', H', \eta'_E, \eta'_H)^\top \in \mathcal{H}$.

We then define the operator $\mathcal{A}$ as follows:

$$\mathcal{D}(\mathcal{A}) = \Big\{ (E, H, \eta_E, \eta_H)^\top \ \in \mathcal{H} |\, \text{curl}\, E, \text{curl}\, H \ \in L^2(\Omega)^3,\ E \times n = 0 \ \text{ on } \Gamma, \\ \partial_s \eta_E \in L^2_w((0,\infty); J(\Omega)), \partial_s \eta_H \in L^2_w((0,\infty); \hat{J}(\Omega)) \\ \text{and } \eta_E(0) = \eta_H(0) = 0 \Big\}\,,$$

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Note that for a field $E \in H(\operatorname{curl}; \Omega) = \{E \in L^2(\Omega)^3 : \operatorname{curl} E \in L^2(\Omega)^3\}$, $E \times n$ has a meaning as an element of $H^{-\frac{1}{2}}(\Gamma)^3$, see [Girault & Raviart (1986)].

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We can prove that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.

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Proof:

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Proof:

We show that $\mathcal{A} - \kappa I$ is a maximal dissipative operator for some $\kappa \geq 0$; then by Lumer-Phillips' theorem it generates a C_0-semigroup of contractions on $\mathcal{H}$ and, consequently, $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.

Boundedness of the semigroup

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$$\Re\left(i\omega\mathcal{L}\nu_E(i\omega)\right) \geq 0, \quad \Re\left(i\omega\mathcal{L}\nu_H(i\omega)\right) \geq 0, \ \forall \omega \in \mathbb{R}. \qquad \textbf{(HP)}$$

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Note that this property is equivalent to

$$\omega\Im\mathcal{L}\nu_E(i\omega) \leq 0, \quad \omega\Im\mathcal{L}\nu_H(i\omega) \leq 0, \ \forall\omega\in\mathbb{R}.$$

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Proof:
Direct consequence of previous proposition and of Theorem 5.2.1 of [Arendt, Batty, Hieber & Neubrander (2001)].

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Strong stability

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Theorem [Arendt & Batty (1988), Lyubich & Vũ (1988)]

Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0 semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and no eigenvalues of A lie on the imaginary axis. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is stable.

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Strong stability

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We now want to take advantage of this theorem. Since the resolvent of our operator is not compact, we have to analyze the full spectrum of $\mathcal{A}$ on the imaginary axis. For that purpose, we actually need a stronger assumption than the passitivity, namely in addition to **(HP)**, we need that

$$\Re\left(i\omega\mathcal{L}\nu_E(i\omega)\right) + \Re\left(i\omega\mathcal{L}\nu_H(i\omega)\right) > 0, \ \forall \omega \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}. \quad \textbf{(HP+)}$$

Strong stability

As before this property is equivalent to

$$\omega \Im \mathcal{L}\nu_E(i\omega) + \omega \Im \mathcal{L}\nu_H(i\omega) < 0, \ \forall \omega \in \mathbb{R}^*.$$

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From this lemma and the theorem of Arendt & Batty / Lyubich & Vũ, we deduce the following result.

PROPOSITION [Nicaise and P., 2020]

Under the assumptions of previous lemma, $(T(t))_{t\geq 0}$ is stable, i.e.,

$$T(t)U_0 \to 0 \text{ in } \mathcal{H}, \text{ as } t \to \infty, \ \forall U_0 \in \mathcal{H}.$$

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In particular the solution $(E(t), H(t))$ of **(P)**, with electric boundary conditions **(BC)** and initial conditions **(IC)** satisfies

$$\|E(t)\|_\Omega + \|H(t)\|_\Omega \to 0 \text{ as } t \to \infty, \ \forall (E_0, H_0) \in J(\Omega) \times \hat{J}(\Omega).$$

Our stability results are based on a frequency domain approach, namely for the exponential decay of the energy we use the following result:

Exponential and polynomial stability results

Our stability results are based on a frequency domain approach, namely for the exponential decay of the energy we use the following result:

Theorem [Pruss 1984 / Huang 1985]

Let $(e^{t\mathcal{L}})_{t\geq 0}$ be a bounded C_0 semigroup on a Hilbert space H. Then it is exponentially stable, i.e., it satisfies

$$||e^{t\mathcal{L}}U_0|| \leq C\, e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$i\mathbb{R} \subset \rho(\mathcal{L}), \qquad (\mathbf{C1})$$

and

$$\sup_{\beta\in\mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty. \qquad (\mathbf{CE})$$

The polynomial decay of the energy is, instead, based on the following result.

Theorem [Borichev and Tomilov, 2010]

Let $(e^{t\mathcal{L}})_{t\geq 0}$ be a bounded C_0 semigroup on a Hilbert space H such that its generator $\mathcal{L}$ satisfies

$$i\mathbb{R} \subset \rho(\mathcal{L}), \qquad (\mathbf{C1})$$

and let ℓ be a fixed positive real number. Then the following properties are equivalent

$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-\frac{1}{\ell}} ||U_0||_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1,$$
$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-1} ||U_0||_{\mathcal{D}(\mathcal{L}^\ell)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^\ell), \quad \forall t > 1,$$
$$\sup_{\beta\in\mathbb{R}} \frac{1}{1+|\beta|^\ell} \|(i\beta - \mathcal{L})^{-1}\| < \infty. \qquad (\mathbf{CP})$$

As we know the validness of assumption **(C1)** , it remains to check whether **(CE)** or **(CP)** is valid. This is possible by improving the assumption **(HP+)** with a precise behavior of $\Re\left(i\omega\mathcal{L}\nu_E(i\omega)\right)$ and of $\Re\left(i\omega\mathcal{L}\nu_H(i\omega)\right)$ at infinity.

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More precisely, we suppose that there exist four non negative constants σ_E, σ_H, ω_0, and m with $\sigma_E + \sigma_H > 0$ such that

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$$\begin{aligned}&\Re\left(i\omega\mathcal{L}\nu_E(i\omega)\right)|X|^2+\Re\left(i\omega\mathcal{L}\nu_H(i\omega)\right)|Y|^2\\&\quad\geq|\omega|^{-m}(\sigma_E|X|^2+\sigma_H|Y|^2),\ \forall X,Y\in\mathbb{C}^3,\ \omega\in\mathbb{R}:|\omega|>\omega_0.\end{aligned}\qquad(\mathbf{HP}++)$$

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$$\Re\left(i\omega\mathcal{L}\nu_E(i\omega)\right)|X|^2+\Re\left(i\omega\mathcal{L}\nu_H(i\omega)\right)|Y|^2 \geq |\omega|^{-m}(\sigma_E|X|^2+\sigma_H|Y|^2),\ \forall X,Y\in\mathbb{C}^3,\ \omega\in\mathbb{R}:|\omega|>\omega_0. \qquad (\mathbf{HP}++)$$

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We have the following result.

PROPOSITION [Nicaise and P., 2020]

In addition to previous assumptions, assume that **(HP++)** holds. Then the operator $\mathcal{A}$ satisfies **(CP)** with $\ell=m$.

From this lemma and the theorem due to Pruss 1984/Huang 1985 (resp. the theorem of Borichev and Tomilov 2010), we deduce the following results.

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COROLLARY

In addition to previous assumptions, assume that **(HP++)** holds with $m = 0$. Then the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is exponentially stable, in particular the solution $(E(t), H(t))$ of **(P)**, with electric boundary conditions **(BC)** and initial conditions **(IC)** tends exponentially to zero in $J(\Omega) \times \hat{J}(\Omega)$.

COROLLARY

In addition to previous assumptions, assume that **(HP++)** holds with $m > 0$. Then the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomially stable, i.e.

$$\|e^{t\mathcal{L}}U_0\| \lesssim t^{-\frac{1}{m}}\|U_0\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1.$$

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In particular the solution $(E(t), H(t))$ of **(P)**, with electric boundary conditions **(BC)** and initial conditions **(IC)** satisfies, $\forall\ t > 1$,

$$\|(E(t), H(t))\|_{J(\Omega)\times\hat{J}(\Omega)} \lesssim t^{-\frac{1}{m}}\|(E_0, H_0)\|_{\mathcal{D}(\mathcal{B})}, \quad \forall (E_0, H_0) \in \mathcal{D}(\mathcal{B}),$$

where

$$D(\mathcal{B}) := \{(E, H) \in J(\Omega)\times\hat{J}(\Omega) |\ \mathrm{curl}\, E, \mathrm{curl}\, H \in L^2(\Omega)^3, E\times n = 0 \text{ on } \Gamma\},$$

is the domain of the operator $\mathcal{B}$ defined by

$$\mathcal{B}(E, H) = (\epsilon E - \mathrm{curl}\, H, \mu H + \mathrm{curl}\, E).$$

Some illustrative examples

All physical examples of dispersive models that we found in the literature (see [Jackson (1962), Kristensson, Rikte & Sihvola (1998), Sihvola (1999), Cassier, Joly & Kachanovska (2017), Bécache, Joly & Vinoles (2018), and Nguyen & Vinoles (2018)]) are summarized in the following example.

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Let J be a positive integer and for all $j \in \{1, \cdots, J\}$, let p_j, q_j be real-valued polynomials (of one variable). Let z_j be a complex number with $\Re z_j = x_j < 0$ and define

$$\nu_E(t) = \sum_{j=1}^{J} (p_j(t)\cos(y_j t) + q_j(t)\sin(y_j t))e^{x_j t},$$

where $y_j = \Im z_j$. Define similarly ν_H by taking other polynomials p_j, q_j and other complex numbers z_j with negative real parts. For simplicity we only examinate the case of ν_E, when necessary we will add the index E or H to distinguish polynomials related to ν_E or ν_H.

Some illustrative examples

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$$\nu_E(t) = \sum_{j=1}^{J} P_j(t) e^{z_j t},$$

where P_j is a (complex-valued) polynomial of degree d_j, we see that

$$\mathcal{L}\nu_E(\lambda) = \sum_{j=1}^{J} \sum_{\ell=0}^{d_j} \frac{P_j^{(\ell)}(0)}{(\lambda - z_j)^{\ell+1}},$$

where $P_j^{(\ell)}$ denotes the derivative of P_j of order ℓ. This means that $i\omega\mathcal{L}\nu_E(i\omega)$ is a rational fraction in ω, more precisely

$$i\omega\mathcal{L}\nu_E(i\omega) = \frac{P_r(\omega)}{Q_r(\omega)} + i\frac{P_i(\omega)}{Q_i(\omega)},$$

where P_r, Q_r, P_i, Q_i are real-valued polynomials such that

$$\deg P_r \leq \deg Q_r \quad \text{and} \quad \deg P_i \leq \deg Q_i.$$

Some illustrative examples

This means that **(HP)** holds if and only if

$$\frac{P_{E,r}(\omega)}{Q_{E,r}(\omega)} \geq 0 \text{ and } \frac{P_{H,r}(\omega)}{Q_{H,r}(\omega)} \geq 0, \ \forall \omega \in \mathbb{R}.$$

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Similarly, **(HP+)** is valid if and only if $R(\omega) = \frac{P_{E,r}(\omega)}{Q_{E,r}(\omega)} + \frac{P_{H,r}(\omega)}{Q_{H,r}(\omega)}$ satisfies

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By writing

$$R(\omega) = \frac{\sum_{n=0}^{N_1} a_n \omega^n}{\sum_{n=0}^{N_2} b_n \omega^n},$$

with $N_1 \leq N_2$, $a_{N_1} \neq 0$ and $a_{N_2} \neq 0$, we notice that two necessary conditions for **(HP+)** are

$$N_2 - N_1 \text{ even and } \frac{a_{N_1}}{b_{N_2}} > 0. \qquad (\star)$$

Some illustrative examples

Finally, the last passitivity assumption **(HP++)** is obviously related to the behavior at infinity of $R(\omega)$. Using previous expression for $R(\omega)$ we deduce that **(HP++)** holds with $m = N_2 - N_1$ if and only if $(\star)$ holds.

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Let us finish by some particular cases.

• The Debye model (cf. [Sihlova (1999)]) corresponds to the choice $\nu_H(t) = 0$ and $\nu_E(t) = \beta e^{-\frac{t}{\tau}}$, with β and τ two positive real numbers.

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$$\mathcal{L}\nu_E(\lambda) = \frac{\beta\tau}{\tau\lambda + 1},$$

and we find

$$R(\omega) = \frac{\beta\tau^2\omega^2}{1 + \tau^2\omega^2}.$$

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This means that **(HP)** and **(HP+)** hold and that **(HP++)** is valid with $m = 0$. Hence, we deduce the exponential decay of the energy if Ω is simply connected with connected boundary.

• The Lorentz model (cf. [Sihlova (1999)]) corresponds to the choice $\nu_H(t) = 0$ and

$$\nu_E(t) = \beta \sin(\nu_0 t) e^{-\frac{\nu t}{2}},$$

with β, ν and ν_0 three positive real numbers. Hence

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with $\omega_0^2 = \nu_0^2 + \nu^2/4$. Then we easily check that **(HP)** and **(HP+)** hold and that **(HP++)** is valid with $m = 2$. Hence, we deduce a decay of the energy as t^{-1} if Ω is simply connected with connected boundary.

Thank you for your attention!