Time-Changed Fractional Ornstein-Uhlenbeck Process

(and its "Fokker-Planck" equation)

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The fractional Ornstein-Uhlenbeck process

In literature there are different (at least three) processes that go under the name of fractional Ornstein-Uhlenbeck process. One of them is the solution of the SDE

$$dU_H(t) = -\frac{1}{\theta}U_H(t)dt + dB_H(t),$$

where $\theta > 0$, $H \in (0,1)$ and $B_H(t)$ is a fractional Brownian motion[1]. Setting $U_H(0) = 0$ for simplicity, the process can be rewritten in terms of a stochastic integral:

$$U_H(t) = e^{-\frac{t}{\theta}} \int_0^t e^{\frac{s}{\theta}} dB^H(s), \; t \geq 0.$$

It is a zero-mean Gaussian process with variance

$$V_{2,H}(t) := \mathbb{E}[(U_H(t))^2] = H\left(\int_0^t \left(e^{-\frac{z}{\theta}} + e^{\frac{1}{\theta}(z-2t)}\right) z^{2H-1}dz\right).$$

[1] Cheridito, Kawaguchi, Maejima, Fractional Ornstein-Uhlenbeck processes, Electron. J. Probab. (2003)

Another fractional OU process

Another kind of fractional OU process is given via a suitable time-change of the classical OU process [2]. Consider an OU process $U(t)$, an α-stable subordinator $\sigma_\alpha(t)$ independent of $U(t)$ and its inverse $L_\alpha(t) = \inf\{y > 0 : \ \sigma_\alpha(y) > t\}$.

The "fractional" OU process is defined as $U^\alpha(t) := U(L_\alpha(t))$.

The name "fractional" in this case refers to the fact that for such process backward and forward equations are identical to the ones of the parent process except for a Caputo derivative in place of the ordinary time derivative.

One can use any subordinator in place of the stable one[3].

[2] Leonenko, Meerschaert, Sikorskii, Fractional Pearson diffusions, J Math Anal Appl (2013)

[3] Gajda, Wyłomańska, Time-changed Ornstein–Uhlenbeck process, J Phys A-Math Theor (2015).

Stationarity and memory

Stationary $U_H(t)$ defined by setting $U_H(0)$ Gaussian with zero mean and variance

$$V_H = \theta^{2H} H\Gamma(2H).$$

- Long-range dependence for $H > 1/2$;
- Short-range dependence for $H < 1/2$.

Stationary $U^\alpha(t)$ defined by setting $U^\alpha(0)$ Gaussian with zero mean and variance

$$V_{\frac{1}{2}} = \theta/2.$$

- First order, but non-second order stationary[4];
- Long-range dependence for any $\alpha \in (0,1)$.

For any subordinator $\sigma(t)$ with inverse $L(t)$, $U^\Phi(t) := U(L(t))$ is stationary if $U(0) \sim \mathcal{N}(0, V_{\frac{1}{2}})$. Short-range dependence is recovered for suitable choices of the Laplace exponent Φ of σ[5].

[4] Leonenko, Meerschaert, Sikorskii, Correlation structure of fractional Pearson diffusions, Comp. Math. Appl. (2013).

[5] Ascione, Leonenko, Pirozzi, Time-Non-Local Pearson Diffusions, J Stat Phys (2021).

Overlapping approaches: the time-changed fOU

Consider a fOU $U_H(t)$ with $U_H(0) = 0$ and a Bernstein function of the form

$$\Phi(\lambda) = \int_0^{+\infty} (1 - e^{-\lambda x})\nu(dx)$$

with ν Lévy measure on $\mathbb{R}^+ := (0, +\infty)$ such that $\nu(\mathbb{R}^+) = +\infty$ and $\int_0^{+\infty}(1 \wedge x)\nu(dx) < +\infty$. Take a subordinator $\sigma(t)$ with Laplace exponent Φ independent of $U_H(t)$ and define its inverse $L(t) = \inf\{y > 0,\ \sigma(y) > t\}$.

The Time-Changed fractional Ornstein-Uhlenbeck process is defined as $U_H^\Phi(t) = U_H(L(t))$[6].

We focus on the case $H > 1/2$, as $H < 1/2$ presents some technical difficulties.

[6] Ascione, Mishura, Pirozzi, Time-changed fractional Ornstein-Uhlenbeck process, Fract. Calc. Appl. Anal. (2020)

Even moments of the time-changed fOU

Define $V_{n,H}(t) := \mathbb{E}[|U_H(t)|^n]$ and $V^{\Phi}_{n,H}(t) := \mathbb{E}[|U^{\Phi}_H(t)|^n]$. By a simple conditioning argument it holds

$$V^{\Phi}_{n,H}(t) = \mathbb{E}[V_{n,H}(L(t))] = \int_0^{+\infty} V_{n,H}(s) f_L(s;t)ds,$$

where $f_L(s;t)$ is the density of $L(t)$. As a direct consequence we have the following result.

Lemma

The functions $V^{\Phi}_{n,H}$ are bounded. Moreover, $V^{\Phi}_{2n,H}$ are increasing and it holds

$$\lim_{t\to+\infty} V^{\Phi}_{2n,H}(t) = \frac{(\theta^{2H}\Gamma(2H+1))^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}.$$

The density of $U_H^\Phi(t)$

Proposition

For any fixed $t > 0$ the r.v. $U_H^\Phi(t)$ is absolutely continuous with density

$$p_H^\Phi(t,x) = \int_0^{+\infty} p_H(s,x) f_L(s;t) ds = \mathbb{E}[p_H(L(t),x)]$$

where $p_H(t,x) = \frac{1}{\sqrt{2\pi V_{2,H}(t)}} e^{-\frac{x^2}{2V_{2,H}(t)}}$ is the probability density function of $U_H(t)$.

It follows from $\mathbb{E}[(L(t))^{-\gamma}] < +\infty$ for any $\gamma \in (0,1)$ and any $t > 0$, $V_{2,H}(t) \sim C(H)t^{2H}$ as $t \to 0$ and

$$\mathbb{E}[e^{izU_H^\Phi(t)}] = \mathbb{E}[e^{-\frac{z^2}{2}V_{2,H}(L(t))}] = \int_0^{+\infty} e^{-\frac{z^2}{2}V_{2,H}(s)} f_L(s;t) ds.$$

The limit distribution

Proposizion[7]

It holds

$$\lim_{t\to+\infty} p_{H,\Phi}(t,x) = \frac{1}{\sqrt{2\pi V_H}} e^{-\frac{x^2}{2V_H}},$$

and then, as $t \to +\infty$, $U_H^{\Phi}(t) \xrightarrow{d} Z$ with $Z \sim \mathcal{N}(0, V_H)$.

This result and the convergence result on even moments were expected as, in a certain sense, $L(t)$ is acting on the time scale of the process.

On the other hand, the convergence of the variance does not directly imply the weak convergence of $U_H^{\Phi}(t)$ due to the fact that the process is not Gaussian anymore.

[7] Ascione, Mishura, Pirozzi, Convergence results for the Time-changed fractional Ornstein-Uhlenbeck processes, Theor. Probab. Math. Stat. (2021), to appear

Dependence on H: one-dimensional convergence

As it is known that $U_H \Rightarrow U$ as $H \to (1/2)^+$ in the space of continuous functions C. What can we say about U_H^Φ?
First: one-dimensional convergence.

Theorem

It holds

$$\lim_{H \to (1/2)^+} p_H^\Phi(t,x) = p^\Phi(t,x), \qquad t > 0,\ x \in \mathbb{R},$$

where $p^\Phi(t,x)$ is the density of $U^\Phi(t)$. Moreover, the convergence is uniform in $\mathbb{R}_0^+ \times K$ (where $\mathbb{R}_0^+ := [0,+\infty)$) for any compact set K such that $0 \notin K$.

Since $x \mapsto |x|^n$ is unbounded, convergence of the moments has to be shown separately.

Proposition

It holds $V_{n,H}^\Phi \to V_{n,1/2}^\Phi$ as $H \to (1/2)^+$ uniformly in $\mathbb{R}_0^+$.

Dependence on H: functional convergence

On the other hand, a continuous mapping approach leads to functional convergence of U_H^Φ.

Theorem

Let D be the space of cádlág functions on $\mathbb{R}_0^+$. Then $U_H^\Phi \Rightarrow U^\Phi$ in D as $H \to (1/2)^+$.

Actually, this result holds even if $\nu(\mathbb{R}^+) < +\infty$ and/or

$$\Phi(\lambda) = a + b\lambda + \int_0^{+\infty} (1 - e^{-\lambda x})\nu(dx).$$

On the other hand, if $a = b = 0$ and $\nu(\mathbb{R}^+) = +\infty$, $L(t)$ is continuous and so do $U_H^\Phi(t)$ and $U^\Phi(t)$. Thus, continuous mapping theorem actually implies that $U_H^\Phi \Rightarrow U^\Phi$ in C.

Differentiability of $V_{2,H}$

We need some further regularity property for the function $V_{2,H}(t)$.

Lemma

It holds $V_{2,H} \in C^1(\mathbb{R}_0^+) \cap L^2(\mathbb{R}^+)$ with

$$\lim_{t\to 0^+} \frac{V'_{2,H}(t)}{2Ht^{2H-1}} = 1 \qquad \lim_{t\to+\infty} \frac{V'_{2,H}(t)}{2H(2H-1)t^{2H-2}e^{-\frac{t}{\theta}}} = 1.$$

Denoting by $\mathcal{L}$ the Laplace transform operator we have:

Lemma

$V'_{2,H}$ admits Laplace transform for any $\lambda \in \mathbb{C}$ with $\Re(\lambda) > -(1/\theta)$ given by

$$\mathcal{L}[V'_{2,H}](\lambda) = \frac{\theta^{2H}\Gamma(2H+1)}{(\theta\lambda+2)(\theta\lambda+1)^{2H-1}}.$$

For any $c > -1/\theta$, the function $\omega \in \mathbb{R} \mapsto \mathcal{L}[V'_{2,H}](c+i\omega)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Weigthed Laplace Transform

For $v:\mathbb{R}_0^+\to\mathbb{R}$ define the weighted Laplace transform operator L_H as

$$L_Hv(\lambda)=\mathcal{L}[V'_{2,H}v](\lambda)=\int_0^{+\infty}V'_{2,H}(t)u(t)e^{-\lambda t}dt.$$

For any $I\subset\mathbb{R}$ we denote by $\mathcal{D}(L_H,I)$ the set of functions $v:\mathbb{R}_0^+\times I\to\mathbb{R}$ such that $L_Hv(\lambda,x)$ is well defined for any $\lambda\in\mathbb{H}:=\{\lambda\in\mathbb{C}:\ \Re(\lambda)>0\}$ and any $x\in I$.

These kind of weighted Laplace transforms generically come into play in the study of time-changed Gaussian processes[8].

[8] Hahn, Ryvkina, Kobayashi, Umarov, On time-changed Gaussian processes and their associated Fokker-Planck-Kolmogorov equations, Electron. Commun. Prob. (2011).

Alternative representation of L_H

Proposition

Fix $c_1 < 0 < c_2$ such that $c_1 - c_2 > -1/\theta$. Let $v : \mathbb{R}_0^+ \to \mathbb{R}$ such that $L_H v(\lambda)$ is well-defined (for instance $v \in L^\infty(\mathbb{R}^+)$) and suppose one of the following hypotheses hold:

a) v is Lipschitz and $\omega \mapsto \mathcal{L}[v](c_2 + i\omega)$ belongs to $L^1(\mathbb{R})$;

b) $v \in L^2(\mathbb{R}^+)$ and $\omega \mapsto \mathcal{L}[v](c_2 + i\omega)$ belongs to $L^2(\mathbb{R})$.

Then, for any $\lambda \in \mathbb{H}$, it holds

$$L_H v(\lambda) = \frac{1}{4\pi^2} \int_0^{+\infty} e^{-\lambda t} \lim_{R_2 \to +\infty} \int_{-\infty}^{+\infty} e^{(c_1 + ix)t}$$

$$\times \int_{-R_2}^{R_2} \frac{\theta^{2H}\Gamma(2H+1)\,\mathcal{L}[v](c_2 + iy)dydxdt}{(\theta(c_1 - c_2 + i(x-y)) + 2)(\theta(c_1 - c_2 + i(x-y)) + 1)^{2H-1}}.$$

The *modified* operator $\widehat{L}_H^{\Phi}$

Let us observe that Φ is well-defined and holomorphic for $\Re(\lambda) > 0$. In particular this means that we can find a vertical line $r_{c_2} = \{c_2 + i\omega,\ \omega \in \mathbb{R}\}$ for some $c_2 > 0$ arbitrarily small such that $\Phi : \Phi^{-1}(r_{c_2}) \to r_{c_2}$ is locally invertible and denote by $\Phi^{-1} : r_{c_2} \to \mathbb{H} := \{\Re(\lambda) > 0\}$ the generic local inverse of Φ. Let $v : \mathbb{H} \to \mathbb{C}$ be such that $\frac{\Phi^{-1}(\lambda)}{\lambda} v(\Phi^{-1}(\lambda))$ is independent of the choice of Φ^{-1} in $\lambda \in r_{c_2}$. Choose then $c_1 < 0$ such that $c_1 - c_2 > -1/\theta$. We define the modified weighted Laplace transform operator $\widehat{L}_H^{\Phi}$ on v as

$$\widehat{L}_H^{\Phi} v(\lambda) = \frac{1}{4\pi^2} \int_0^{+\infty} e^{-\lambda t} \lim_{R_2 \to +\infty} \int_{-\infty}^{+\infty} e^{(c_1+ix)t}$$

$$\times \int_{-R_2}^{R_2} \frac{\theta^{2H}\Gamma(2H+1)\,\mathcal{L}[v](\Phi^{-1}(c_2+iy))\Phi^{-1}(c_2+iy)dydxdt}{(c_2+iy)(\theta(c_1-c_2+i(x-y))+2)(\theta(c_1-c_2+i(x-y))+1)^{2H-1}}.$$

We don't really need vertical lines: since we are working with holomorphic function, we could use any $\mathcal{C}$ homeomorphic to r_{c_2} in $\mathbb{H}$.

Subordination and weighted subordination

For any Banach space X, define the subordination and weighted subordination operators $S^{\Phi}, S^{\Phi}_H : L^{\infty}(\mathbb{R}^+; X) \to L^{\infty}(\mathbb{R}^+; X)$ as

$$S^{\Phi} v(t) = \mathbb{E}[v(L(t))] \qquad S^{\Phi}_H v(t) = \mathbb{E}[v(L(t)) V'_{2,H}(L(t))].$$

Proposition[9]

The operators S^{Φ} and S^{Φ}_H are continuous with $\|S_{\Phi}\| \le 1$ and $\|S_{\Phi,H}\| \le \left\| V'_{2,H} \right\|_{L^{\infty}(\mathbb{R}^+)}$, injective and, for any $\lambda \in \mathbb{H}$, it holds

$$\mathcal{L}[S^{\Phi} v](\lambda) = \frac{\Phi(\lambda)}{\lambda} \mathcal{L}[v](\Phi(\lambda)), \quad \mathcal{L}[S^{\Phi}_H v](\lambda) = \frac{\Phi(\lambda)}{\lambda} L_H v(\Phi(\lambda)).$$

[9] Ascione, Mishura, Pirozzi, The Fokker-Planck equation for the time-changed fractional Ornstein-Uhlenbeck stochastic process, Proc. Roy. Soc. Edinb. A (2021), to appear

Subordinated functions and the operator $\widehat{L}_H$

For subordinated functions, one can give a more explicit link between the operators L_H and $\widehat{L}_H^{\Phi}$.

Proposition

Suppose $v_\Phi = S^\Phi v$ for some $v \in L^\infty(\mathbb{R}^+; X)$ satisfying hypotheses a) or b). Then it holds

$$\widehat{L}_H^{\Phi}(\mathcal{L}[v_\Phi])(\lambda) = L_H v(\Phi(\lambda)).$$

This is the case of $p_H^\Phi(t,x)$ as $p_H^\Phi(t,x) = S^\Phi p_H(t,x)$ and $p_H(\cdot,x)$ is Lipschitz and bounded if $x \neq 0$ with Laplace transform integrable on vertical lines.

The generalized Caputo-type derivative

Let us give the definition of generalized Caputo-type derivative[10,11]. Given a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+;X)$ with $u(0+) \in \mathbb{R}$ we define the generalized Caputo-type derivative as

$$\frac{d^\Phi}{dt^\Phi}u(t) = \frac{d}{dt}\int_0^t \bar{\nu}(t-\tau)(u(\tau)-u(0+))d\tau,$$

where $\bar{\nu}(t) = \nu(t,+\infty)$, provided the involved quantities are well-defined. If u is absolutely continuous, then $\frac{d^\Phi}{dt^\Phi}u(t)$ is well defined (in $L^1_{\mathrm{loc}}(\mathbb{R}^+;X)$) and

$$\frac{d^\Phi}{dt^\Phi}u(t) = \int_0^t \bar{\nu}(t-\tau)u'(\tau)d\tau.$$

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[10] Kochubei, General fractional calculus, evolution equations, and renewal processes, Integr. Equat. Oper. Th. (2011).

[11] Toaldo, Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups, Potential Anal. (2015).

The Fokker-Planck operator

On the other hand, we have all the ingredients to define a "Fokker-Planck" operator for U_H^Φ.
For a function $v : I \times \mathbb{R}^+ \to \mathbb{R}$ where $I \subset \mathbb{R}$ we define

$$\mathcal{F}_H^\Phi v(x,t) = \mathcal{L}^{-1}\left[\frac{\Phi(\lambda)}{\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_H^\Phi(\mathcal{L}[v(x,\cdot)])\right](t),$$

provided the involved quantities are well-defined.

If $\Phi(\lambda) = \lambda$, then $\mathcal{F}_H^\Phi = V'_{2,H}(t)\frac{\partial^2}{\partial x^2}$, as expected, since we should recover the Fokker-Planck equation of $U_H(t)$ as a Gaussian process.

On the other hand, if $H = 1/2$, $\mathcal{F}_H^\Phi$ does not coincide with the generator of the OU process.

The Fokker-Planck equation

We say that a function $v : I \times \mathbb{R}_0^+ \to \mathbb{R}$ is a *solution* of

$$\frac{\partial^\Phi}{\partial t^\Phi} v(x,t) = \frac{1}{2} \mathcal{F}_H^\Phi v(x,t) \qquad (x,t) \in I \times \mathbb{R}^+ \tag{FP}$$

if $v \in D(\mathcal{F}_H^\Phi, I)$, $\partial_t^\Phi v(x,\cdot)$ is well-defined for any $x \in I$ and Equation (FP) holds pointwise a.e.

We say that v is a *strong solution* of Equation (FP) if it is a solution, it is continuous as a function of two variables in $\overline{I} \times \mathbb{R}_0^+$ and, for any $x \in I$, $v(x,\cdot) \in C^1(\mathbb{R}^+) \cap W^{1,1}(0,1)$.

p_H^Φ and the Fokker-Planck equation

Theorem

p_H^Φ is solution of Equation (FP) on $\mathbb{R}^* \times \mathbb{R}^+$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Suppose Φ is regularly varying at infinity with index $\gamma \in (0,1)$. Then, if p_H^Φ is a strong solution of Equation (FP), it is the unique solution subject to the initial/boundary/limit conditions

$$\begin{cases} \lim_{x \to \pm\infty} p_H^\Phi(x,t) = 0 & \text{locally uniformly} \\ p_H^\Phi(0,t) = \int_0^{+\infty} p_H(0,s) f_L(s;t) ds & t > 0 \\ p_H^\Phi(x,0) = 0 & x \in \mathbb{R}^* \end{cases}$$

If $\Phi(\lambda) = \lambda^\alpha$ for some $\alpha \in (0,1)$, then it is not difficult to show that p_H^Φ is a strong solution of (FP).

Weak Maximum principle

Theorem

Let Φ be regularly varying at infinity of index $\gamma \in (0,1)$. Let $v_\Phi = S^\Phi v$ be a strong solution of (FP) in $(a,b) \times \mathbb{R}^+$ with v satisfying hypotheses a) or b). Fix $T > 0$ and define $\mathcal{O} = (a,b) \times (0,T]$. Let $\partial_p \mathcal{O}$ be its parabolic boundary. Then

$$\max_{(x,t)\in\mathcal{O}} v_\Phi(x,t) = \max_{(x,t)\in\partial_p\mathcal{O}} v_\Phi(x,t).$$

Corollary

If Φ is regularly varying at infinity of index $\gamma \in (0,1)$ and $v_\Phi = S^\Phi v$ and $w_\Phi = S^\Phi w$ are two strong solutions of (FP) in $(a,b) \times \mathbb{R}^+$ with v, w satisfying hypotheses a) or b) and such that $w_\Phi(x,0) = v_\Phi(x,0)$ for $x \in (a,b)$, $w_\Phi(a,t) = v_\Phi(a,t)$ and $w_\Phi(b,t) = v_\Phi(b,t)$ for $t > 0$, then $v_\Phi \equiv w_\Phi$ in $[a,b] \times \mathbb{R}_0^+$.

Mild Solutions

We say that v is a *mild solution* of Equation (FP) iff it is Laplace transformable in the time variable for any $x \in I$, its Laplace transform belongs to the domain of $\widehat{L}$, $x \in I \mapsto \mathcal{L}[v(x,\cdot)](\lambda)$ is continuous for any fixed $\lambda \in \mathbb{H}$ and the following equality holds for any $x \in I$ and $\lambda \in \mathbb{H}$:

$$\Phi(\lambda)\,\mathcal{L}[v(x,\cdot)](\lambda) - \frac{\Phi(\lambda)}{\lambda}v(x,0) = \frac{\Phi(\lambda)}{2\lambda}\frac{\partial^2}{\partial x^2}\widehat{L}_H^{\Phi}(\mathcal{L}[v(x,\cdot)])(\lambda). \quad \text{(mFP)}$$

Recalling that $\widehat{L}_H^{\Phi} = L_H$ if $\Phi(\lambda) = \lambda$, the same definition of mild solution holds in this case.

Proposition

Let $v_\Phi = S^\Phi v$ with v satisfying hypotheses a) or b). Then v_Φ is mild solution of (FP) if and only if v is mild solution of (FP) with $\Phi(\lambda) = \lambda$.

Gain of regularity for mild solutions

Proposition

Let $v \in L^\infty(\mathbb{R}^+; C^0(I))$ such that hypotheses a) or b) hold, $v_\Phi = S^\Phi v$ be a mild solution with $v_\Phi \in D(\mathcal{F}_H^\Phi, I)$ and $\mathcal{F}_H^\Phi v_\Phi(\cdot, t) \in C^0(I)$ for any fixed $t > 0$. Then $S_H^\Phi v(\cdot, t) \in C^2(I)$ and

$$\mathcal{F}_H^\Phi v_\Phi(x,t) = \frac{\partial^2}{\partial x^2} S_H^\Phi v(x,t).$$

Theorem

Let $v \in L^\infty(\mathbb{R}^+; C^0(I))$ such that hypotheses a) or b) hold, $v_\Phi = S^\Phi v$ be a mild solution of Equation (FP). Suppose that $V'_{2,H}(\cdot)\frac{\partial^2}{\partial x^2} v(x,\cdot) \in L^\infty(\mathbb{R}^+)$ for any fixed $x \in I$. Then v_Φ is a classical solution of (FP).

If $\Phi(\lambda) = \lambda^\alpha$ for $\alpha \in (0,1)$ and v satisfies hypothesis a), then v_Φ is strong solution of (FP).

Dependence on H: convergence to a solution

Theorem

Consider a sequence $H_n \to (1/2)^+$ with $H_n \in (1/2,1)$ and $v_\Phi^{(n)} = S^\Phi v^{(n)}$ for some $v^{(n)} \in L^\infty(\mathbb{R}^+; C^2(I))$ satisfying a) or b). Suppose $v_\Phi^{(n)}$ is a mild solution of (FP) (with $H = H_n$) and that there exists a function $v \in L^\infty(\mathbb{R}^+; C^2(I))$ such that $v^{(n)} \to v$ in $L^\infty(\mathbb{R}^+; C^2(I))$. Then $v_\Phi = S^\Phi v$ is mild solution of (FP) for $H = 1/2$.
If, additionally, $V'_{2,H_n} \frac{\partial^2}{\partial x^2} v^{(n)}(\cdot, x) \in L^\infty(\mathbb{R}^+)$, then v_Φ is a classical solution of (FP) for $H = 1/2$.

Future plans

- Extending the results to the case $H < 1/2$. In such case $V_{2,H}$ is only absolutely continuous and $V'_{2,H}(t) \to +\infty$ as $t \to 0^+$: is it possible to obtain the FP equation in this setting?
- Studying the covariance function (for any $H \in (0, 1)$): what happens to the long-range/short-range dependence when we overlap the two different kind of memory?

Thank you for the attention!!!