Graph Theory Between The World Wars

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Discrete mathematics in the interwar period

- Combinatorics was already a well-established mathematical discipline, new results published in respected journals (Math. Ann., J. Lond. Math. Soc., Amer. J. Math.)
- Graph theory had a bad reputation as a science of trivial problems, new results often formulated in the language of matrices or topological structures; first textbook published in 1936
- Key results were often discovered while investigating other topics (set theory, topology), some of them were motivated by practical problems (e.g., construction of electricity networks)

Four-color problem and graph coloring (1)

The four color conjecture was proposed in 1852 by Francis Guthrie. Although the 1879 proof by Alfred Kempe turned out to be incorrect, his idea of reducible configurations and unavoidable sets of configurations finally led to the computer-assisted proof of the conjecture in 1976 by Kenneth Appel and Wolfgang Haken.

In the meanwhile, some important results were obtained by Philip Franklin. In 1922, he discovered new reducible configurations, and showed that every irreducible map must contain more than 25 regions. Thus, he verified the four color conjecture for each map containing at most 25 regions.

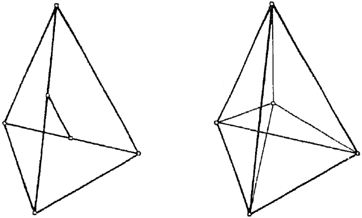
Four-color problem and graph coloring (2)

George D. Birkhoff (1912): $P(\lambda)$ = the number of ways of coloring the map in λ colors. P is a polynomial of degree n, where n is the number of regions – the chromatic polynomial of a given map. Birkhoff found a formula for the coefficients of P. The four-color problem is equivalent to $P(4) > 0$ for all maps.

Hassler Whitney (1932) – dissertation on graph coloring supervised by Birkhoff; extension of chromatic polynomial to all graphs, a simple proof of formulas for the coefficients; it suffices to study the four-color problem for Hamiltonian graphs

Characterization of planar graphs

- Kazimierz Kuratowski (*Sur le problème des courbes gauches en Topologie*, 1930): a graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$; more general formulation in terms of topological notions (continuum)



- Hassler Whitney (1932, 1933): definition of an abstract dual graph; a graph is planar if and only if it has an abstract dual

Menger's theorem and Cayley's formula

Menger's theorem:

- Karl Menger (*Zur allgemeinen Kurventheorie*, 1927): Let A, B be two disjoint sets of vertices. Then there exist n vertex-disjoint paths between A and B if and only if A, B cannot be separated by deleting n vertices.
- Menger's comment (1981): *Some graph theorists may be surprised to learn that this graph theoretical assertion first came up in 1926 as a lemma in proving an extremely general theorem of set theoretical curve theory.*

Cayley's formula:

- Arthur Cayley (1889): The number of trees on n labeled vertices is n^{n-2}; rigorous proof is missing.
- Heinz Prüfer (*Neuer Beweis eines Satzes über Permutationen*, 1918): Bijection between trees and sequences of numbers from $\{1, \ldots, n\}$ of length $n-2$. (*In how many ways is it possible to connect n cities using a railroad network?*)

Minimum spanning tree (1)

Given a connected weighted graph, find its minimum spanning tree.

The first algorithm for solving this problem was proposed by the Czech mathematician Otakar Borůvka in 1926. The problem was suggested to Borůvka during World War I by his friend Jindřich Saxel, who worked for the West-Moravian Powerplants, and stated the problem in terms of cities and the distances between them.

Borůvka reformulated the whole problem in the language of matrices, whose elements correspond to edge weights. His description of the algorithm for finding the optimal solution is long and complicated, but becomes much more transparent when reformulated in graph-theoretical language:

Begin by joining each vertex with its nearest neighbor, obtaining a certain forest. For each component, add the shortest edge joining it to a different component, and repeat this step until obtaining a connected graph.

Minimum spanning tree (2)

An alternative algorithm for solving the same problem was proposed by the Czech mathematician Vojtěch Jarník in 1930.

The problem was again formulated and solved in an algebraic way. The algorithm in graph-theoretical language is as follows:

Begin with an arbitrary vertex, and find the shortest edge incident with this vertex, giving rise to a tree with two vertices. Add the shortest edge joining the tree to a vertex that is not included in the tree. Repeat this step until obtaining a tree containing all vertices.

The algorithm is often called Prim's algorithm after Robert Clay Prim, who discovered it independently in 1957.

Dénes Kőnig – pioneer of graph theory

THEORIE DER ENDLICHEN
UND UNENDLICHEN GRAPHEN

KOMBINATORISCHE TOPOLOGIE DER STRECKENKOMPLEXE

VON

DÉNES KÖNIG

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MIT 107 FIGUREN

LEIPZIG 1936
AKADEMISCHE VERLAGSGESELLSCHAFT M. B. H.

Author of first textbook in graph theory: *Theorie der endlichen und unendlichen Graphen* (1936, republished 1950, 1986); English translation *Theory of Finite and Infinite Graphs* (1990)

Some distinctive features of the textbook

- Rigorous foundations and proofs
- Coverage of undirected and directed graphs, multigraphs, finite as well as infinite graphs
- Applications (set theory, linear algebra, recreational mathematics)
- Historical and bibliographic remarks

Set theory and infinite graphs

Dénes Kőnig was interested in his father's Gyula (Julius) Kőnig research in set theory, and realized that some results might be more transparent in the language of infinite graphs.

Cantor-Bernstein-Schröder theorem: If there exist injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

Kőnig's graph-theoretic proof: Assume that A and B are disjoint. Consider bipartite graph G with parts A and B. Vertices $x \in A$, $y \in B$ are joined by an edge iff $f(x) = y$ or $g(y) = x$ (take two edges if both statements hold). It suffices to show the existence of a perfect matching in G.

Show that each component of G is either a cycle of even length, a singly infinite path, or a doubly infinite path. Hence, there is a perfect matching in every component, and therefore also a perfect matching in the whole graph, giving rise to a bijection $h : A \to B$.

The status of graph theory in the 1930s

Paul Erdős in 1977:

I cannot help feeling sorry that Dénes Kőnig did not live to see the present flowering of graph theory to which he contributed so much. It is curious how little graph theory and combinatorial analysis was appreciated in those early dark ages. . . . When I first got to Princeton in 1938, I was surprised how many of the topologists looked down upon the four colour problem and considered it an unimportant side issue.

William Thomas Tutte in the introduction to the English translation of Kőnig's textbook:

But the honour of presenting Graph Theory to the mathematical world as a subject in its own right, with its own textbook, belongs to Dénes Kőnig. Low was the prestige of Graph Theory in the Dirty Thirties. . . . Students tempted by Graph Theory would be advised by their supervisors to turn to something respectable or even useful, like differential equations. I am reminded that my own most recent research in Graph Theory has involved differential equations. Mathematics is One, after all.

References

- R. Wilson, J. J. Watkins (eds.): *Combinatorics: Ancient and Modern*, Oxford University Press, 2013
- N. L. Biggs, E. K. Lloyd, R. J. Wilson: *Graph Theory 1736–1936*, Oxford University Press, 1998
- A. Slavík, *Transformation of Mathematics Between World Wars: The Case of Combinatorics*, in M. Bečvářová (ed.), *The Development of Mathematics Between The World Wars*, World Scientific, 2021, 549–586