The 8th European Congress of Mathematics (8ECM)
Minisymposium “Graphs, Polynomials, Surfaces, and Knots” (MS - ID 49)
Online (Zoom), Tuesday, 22 June 2021
18:00 CEST (19:00 MSK)

A new enumerator polynomial with a smart derivative

Serge Lawrencenko

- **Ω** : a given set of unlabeled substructures of some ambient discrete structure S.
- **Ω_*k*** : a subset of Ω that consists of *k*-symmetric substructures.
- **Λ** : obtained from Ω by labeling the members of Ω in all combinatorially different ways.
- **Γ** : the automorphism group of S, which naturally acts on Λ.
- The enumerative polynomial *M*(*x*) is defined by:

$$M(x) = \sum_{k \mid |\Gamma|} |\Omega_k| x^{k-1}$$

Properties of the enumerative polynomial *M*(*x*)

$$M(x) = \sum_{k \mid |\Gamma|} |\Omega_k| x^{k-1}$$

the enumerative polynomial.

$$M(1) = |\Omega|$$

The total number of unlabeled structures.

$$M'(1) = \tilde{\alpha}$$

the total number of non-trivial automorphisms of substructures over Ω.

$$\int_0^1 M(x)\,dx = \frac{|\Lambda|}{|\Gamma|}$$

the quotient of the number of labeled structures and the order of the acting group.

Example: Trees on 4 vertices

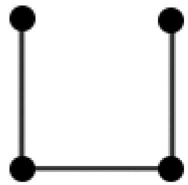
• Ω : the set of (vertex-) unlabeled trees (that is, up to isomorphism), as subgraphs of the ambient complete graph S = K_4.

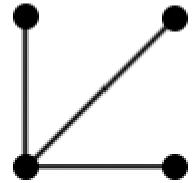
• Ω_k : the set of unlabeled k-symmetric trees with 4 vertices.

• Λ : the set of all labeled trees with 4 vertices.

• Γ : the automorphism group of the complete graph S = K_4, acting on Λ.

Example: Trees on 4 vertices





$$M(x) = x + x^5, \qquad M(1) = 1 + 1 = 2$$

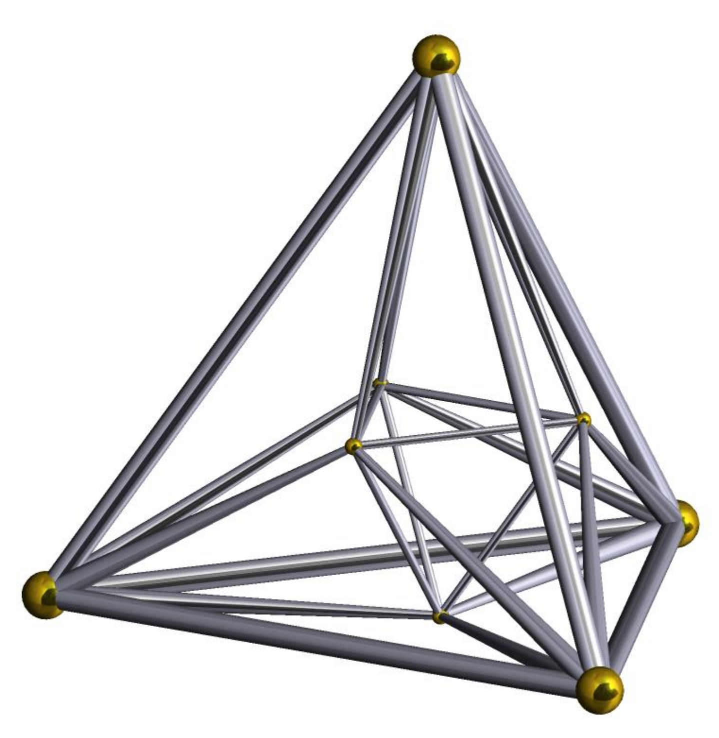
$$\int_0^1 M(x)\,dx = \frac{1}{6} + \frac{1}{2} = \frac{2}{3} = \frac{|\Lambda|}{|\Gamma|} = \frac{|\Lambda|}{4!} = \frac{|\Lambda|}{24}$$

$$|\Lambda| = \frac{2 \cdot 24}{3} = 16 = 4^{4-2}$$

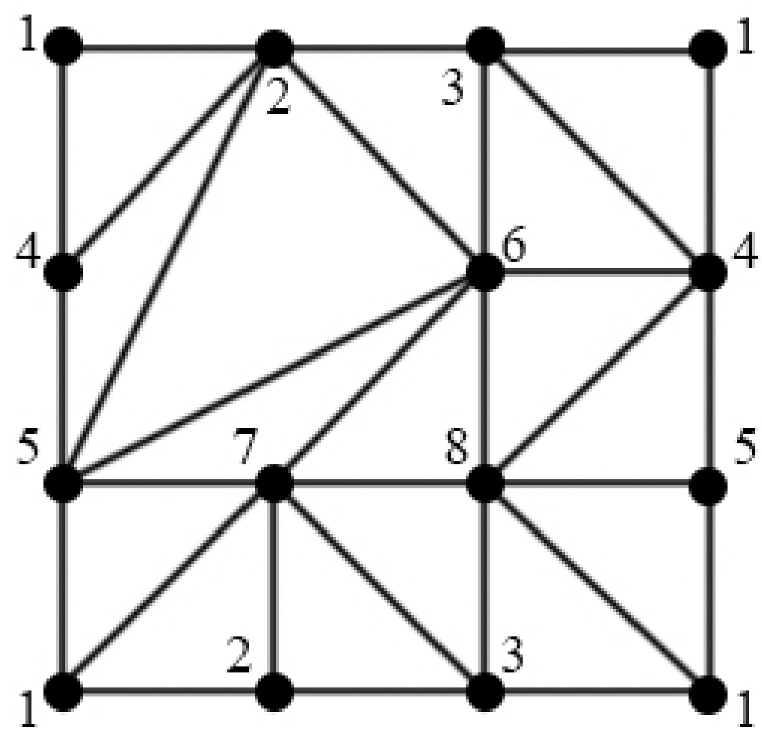
-- the total number of labeled trees on 4 vertices,

which agrees with Cayley's formula.

G : Graph *K*_{2, 2, 2, 2} :
the complete 4-partite graph
with 8 vertices,
the 1-skeleton of the 16-cell,
one of the six regular convex
4-polytopes.



T : a 6-regular triangulation
of the torus with the graph *G*
(6-regular triangulation with 8 vertices).



Vertex-labeled triangulations with the graph G

- Ω : the set of (vertex-) unlabeled toroidal triangulations with graph ***G*** (up to isomorphism).

- Ω_k : the subset of Ω made up of *k*-symmetric triangulations.

- Λ : the set of all labeled triangulations with the graph ***G***,

  geometrically: Λ is the set of toroidal geometric 2-subcomplexes of the 16-cell.

- Γ : the automorphism group of the graph ***G*** = $K_{\{2, 2, 2, 2\}}$ acting on Λ.

Vertex-labeled triangulations with the graph G

It is known that Ω = {***T***}, |Ω| = |Ω_32| = 1.

$$M(x) = x^{31}$$: the enumerative polynomial.

- Λ : the set of all labeled triangulations of the torus with the graph ***G***.

- Γ : the automorphism group of the graph ***G*** = K_{2, 2, 2, 2} acting on Λ.   We have |Γ| = 384.

$$\int_0^1 M(x)\,dx = \frac{1}{32} = \frac{|\Lambda|}{|\Gamma|} = \frac{|\Lambda|}{384}$$

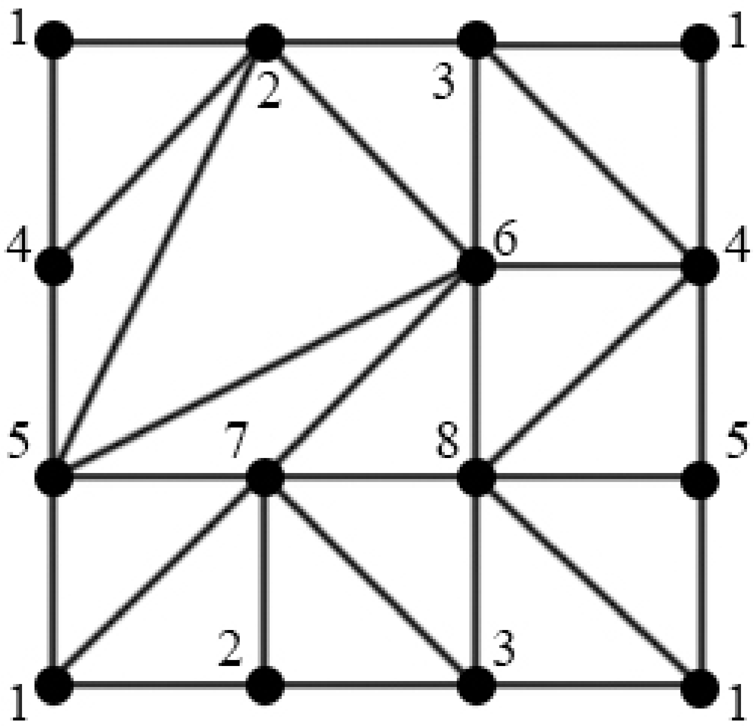
$$|\Lambda| = \frac{384}{32} = 12$$ -- the total number of labeled triangulations with graph ***G***.

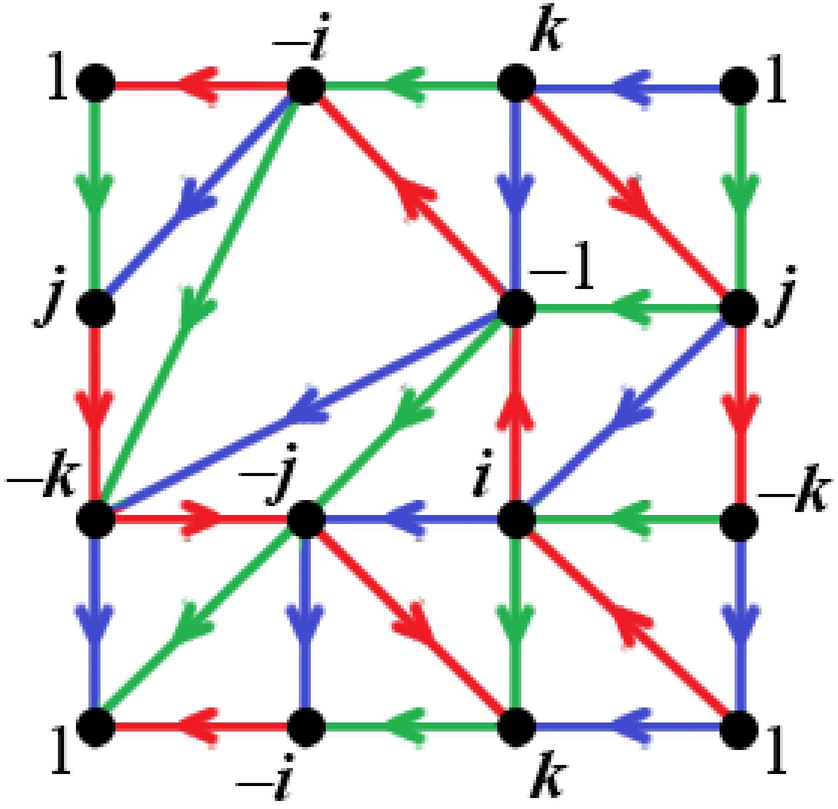
Problem

Determine explicitly all the members of Λ,
that is, all pairwise different labeled triangulations
with the graph ***G***,

without computer aid.

The key idea to solve the Problem: Change ***G*** to ***G****, the Cayley graph of the quaternion group Q_8:

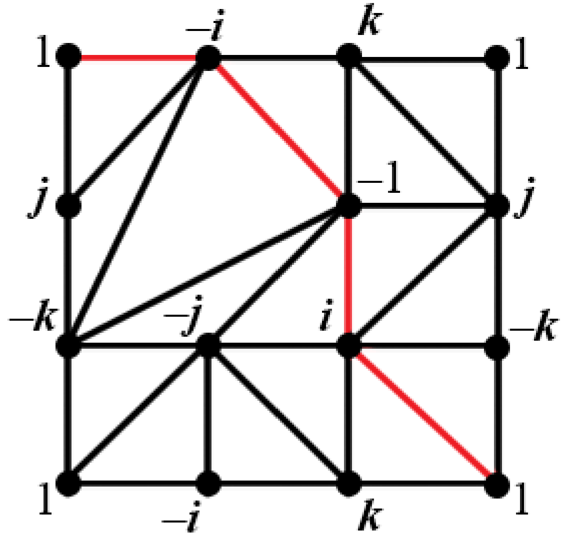




Red arcs = multiplication by ***i*** (on the right),
Green arcs = multiplication by ***j*** ,
Blue arcs = multiplication by ***k*** .

T^*

$D_8(1, i, -1, -i)/Z(D_8)$



1
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k
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1
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k
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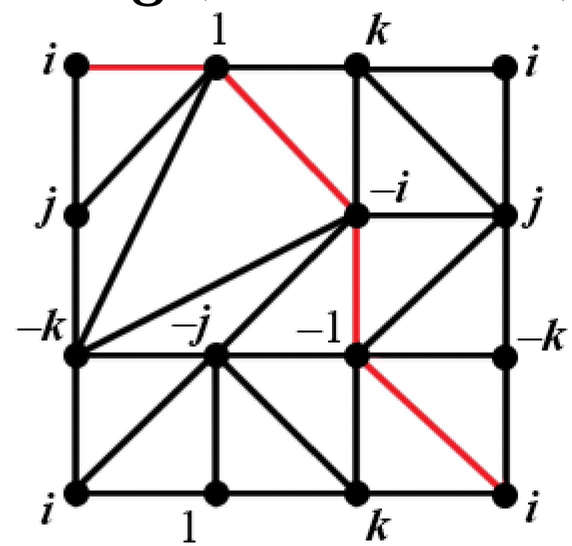

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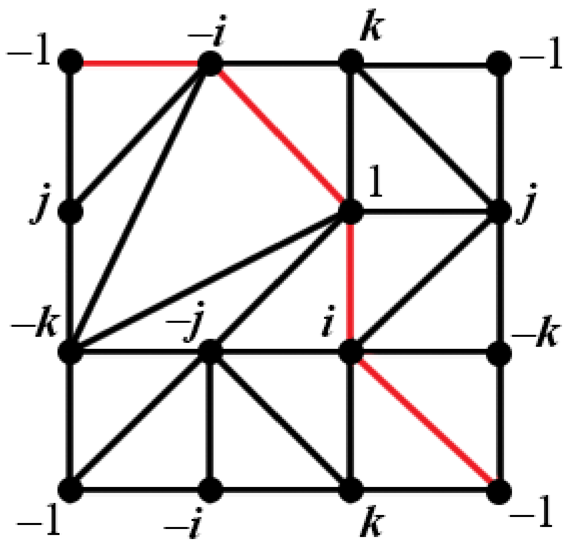

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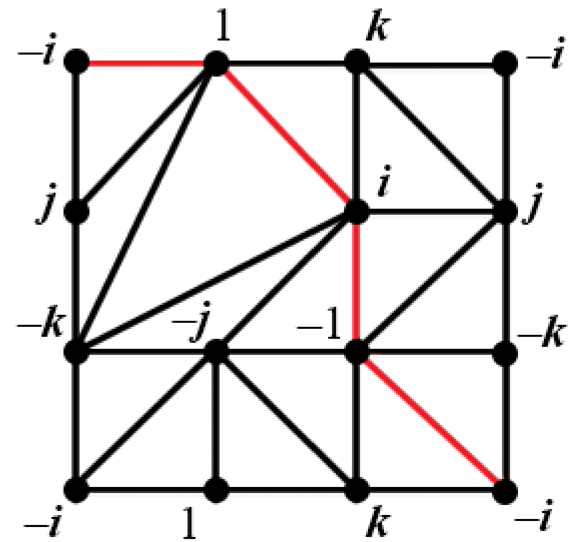
−1



−i
1
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−j
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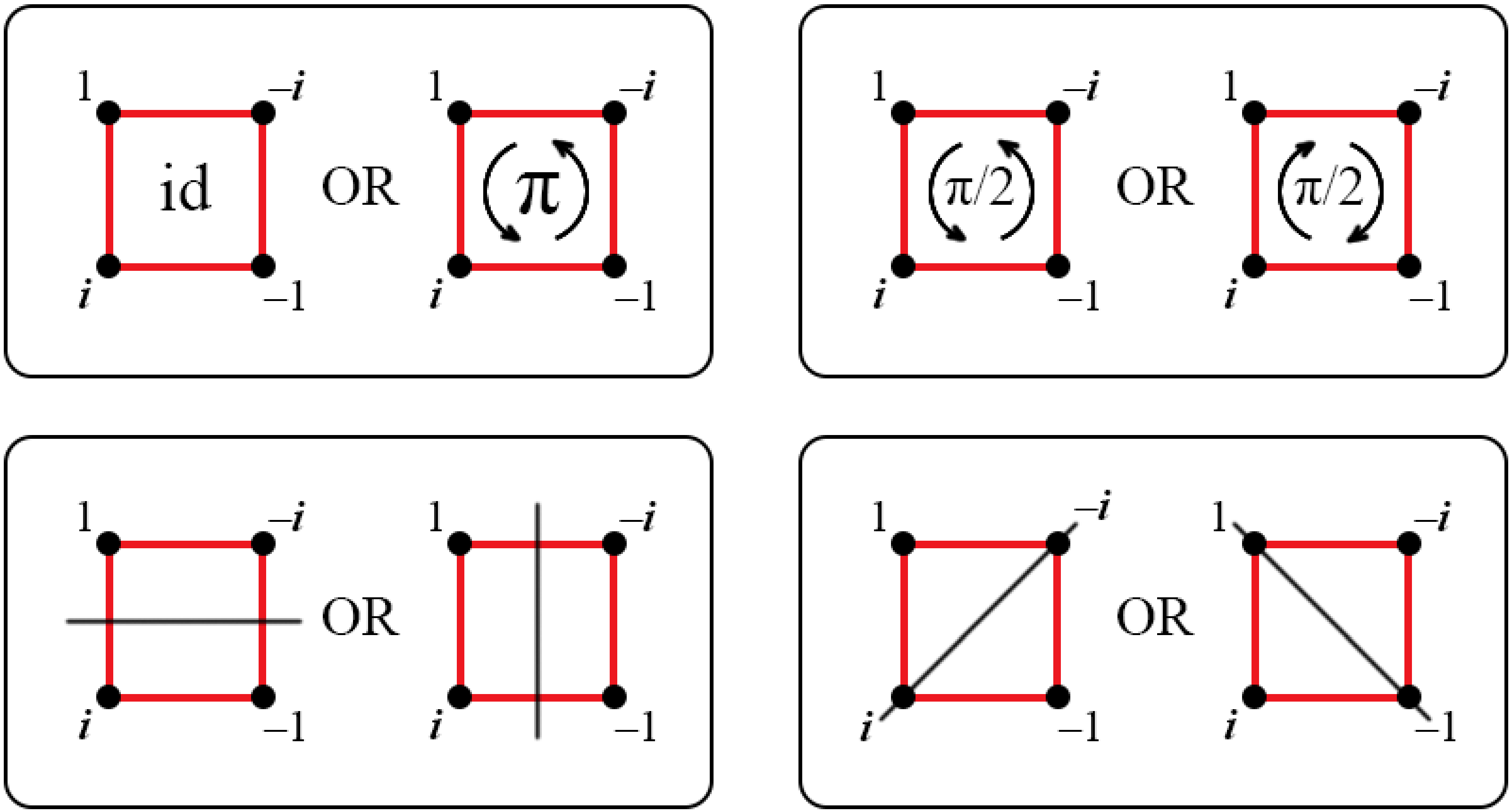
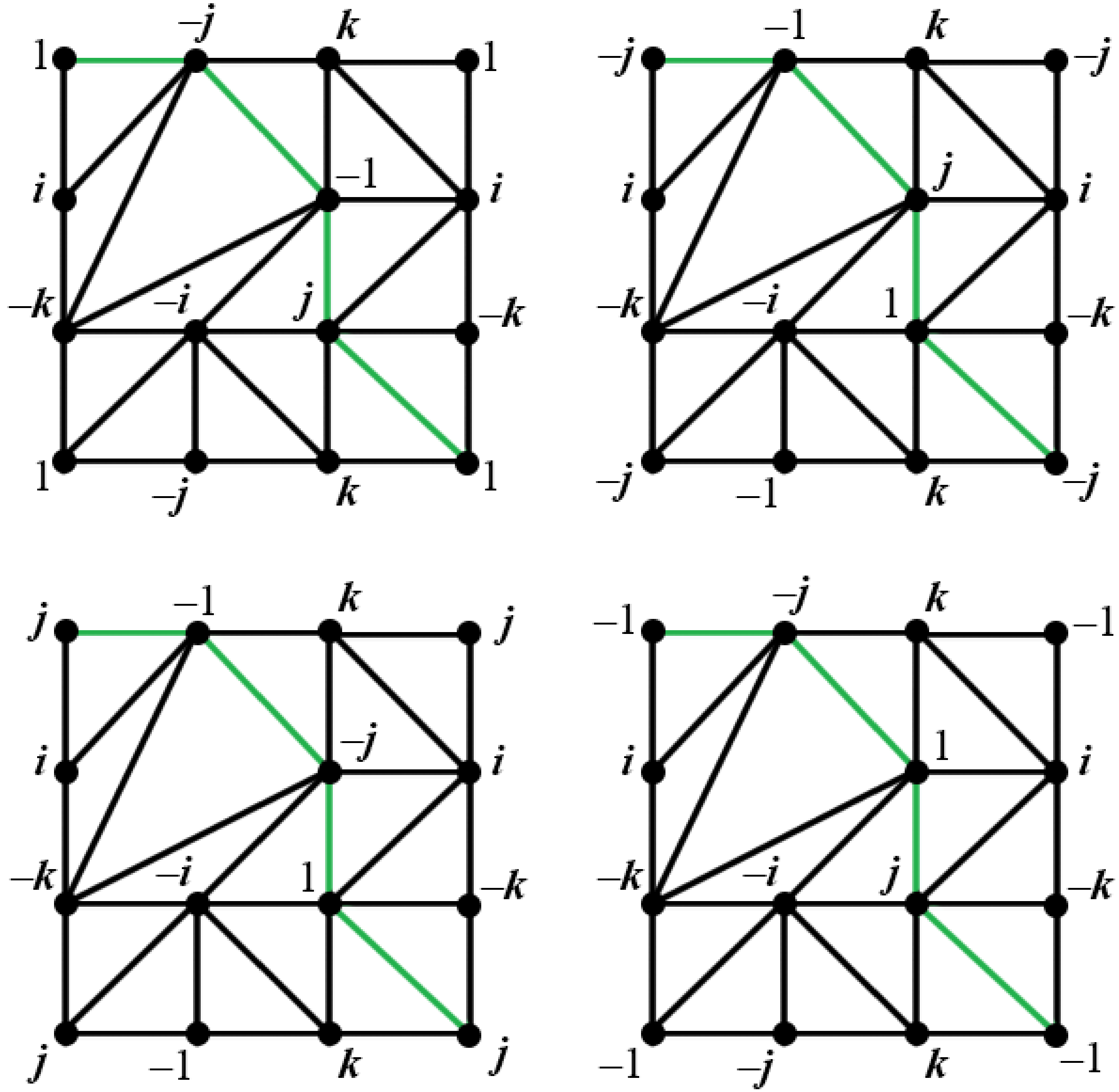
−k
−i
1
k
−i

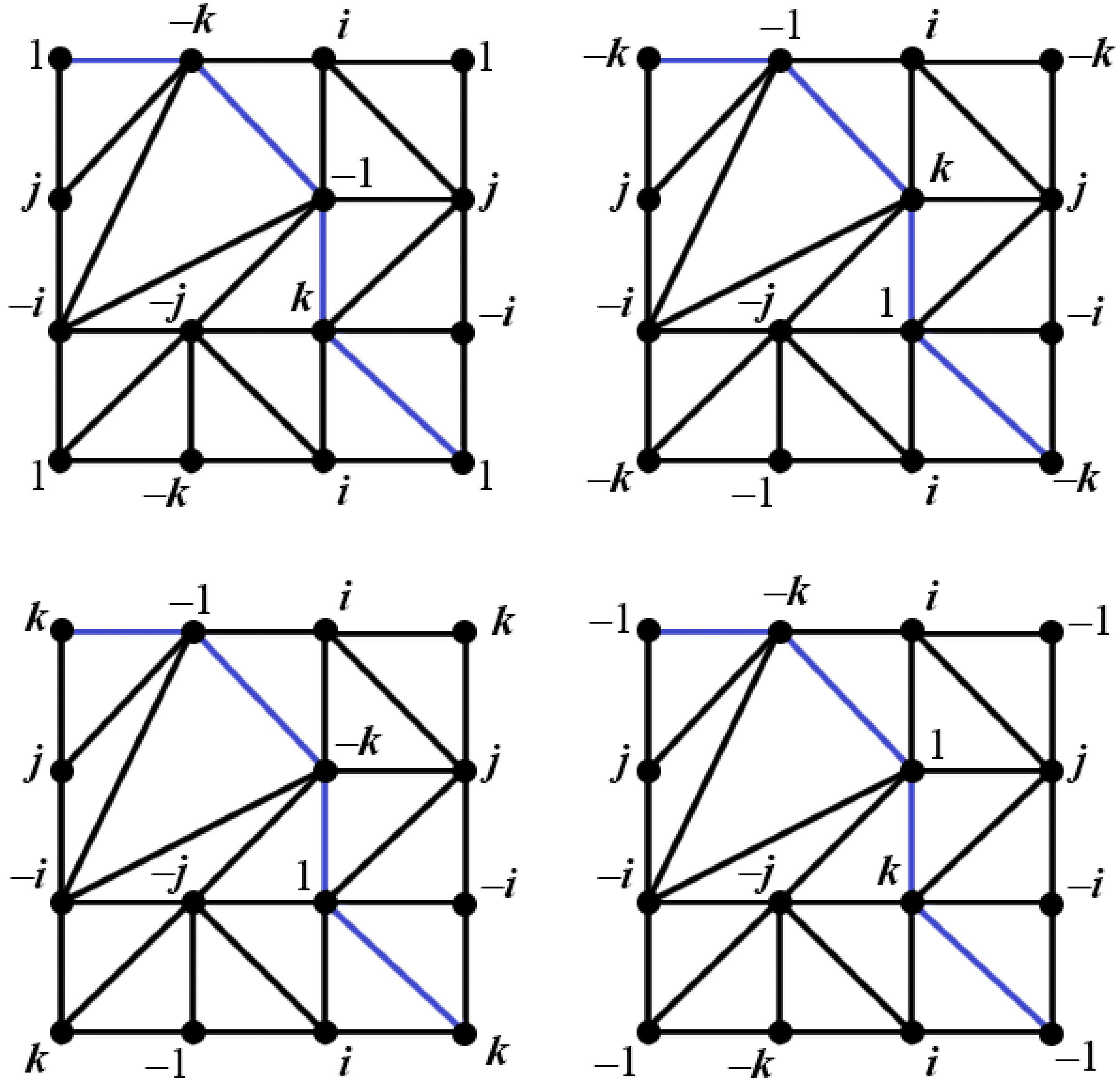
$$D_8(1, i, -1, -i)/Z(D_8)$$



$(i\ j)(-i\ -j)T^*$ $\qquad$ $D_8(1, j, -1, -j)/Z(D_8)$

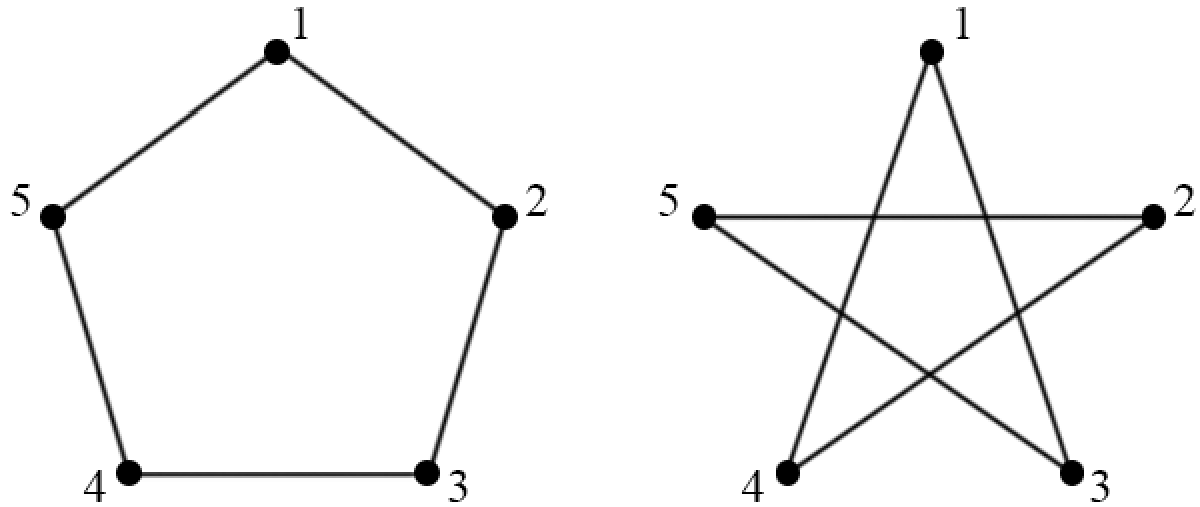


$(i\ k)(-i\ -k)T^*$ $\qquad$ $D_8(1,k,-1,-k)/Z(D_8)$



Corollary

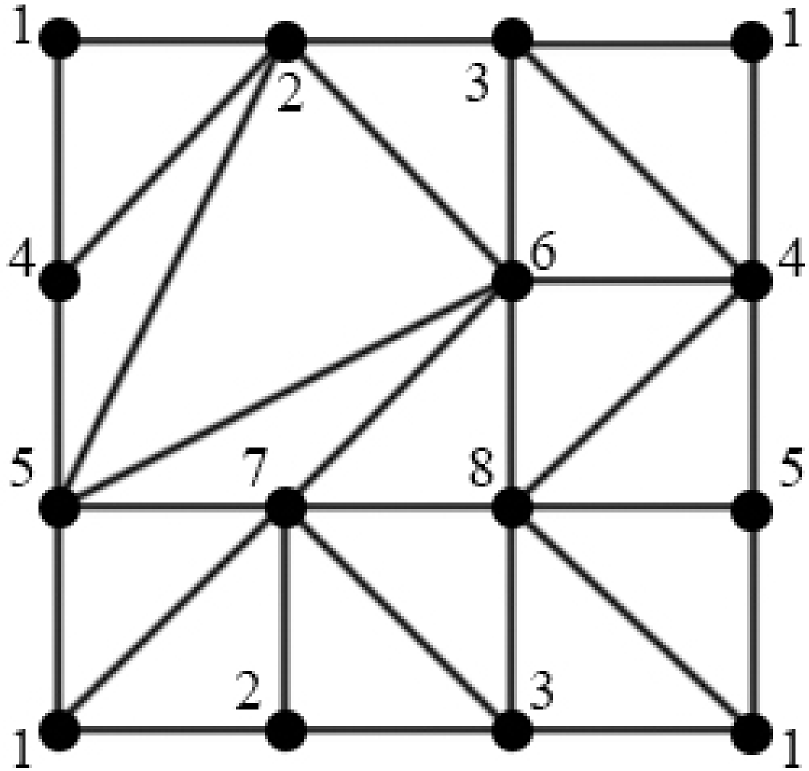
The triangulation ***T*** is the only, up to isomorphism, self-complementary 2-complex homeomorphic to the 2-torus. Furthermore, **there exist, in all, 6 pairs of mutually complementary labeled simplicial 2-complexes homeomorphic to the 2-torus, which have as underlying 2-complex the triangulation *T* (6-regular triangulation of the 2-torus with 8 vertices).**

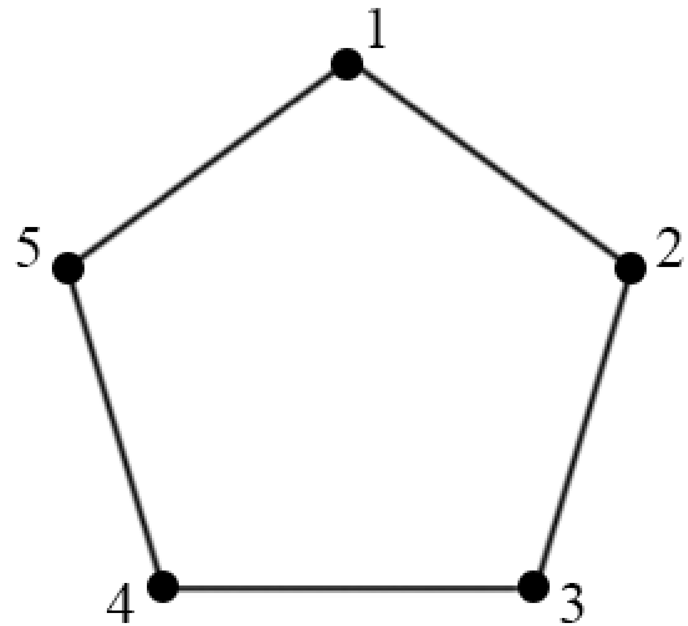


Remark

It is not hard to verify that the cycle C_5 is the only,
up to isomorphism,
self-complementary graph homeomorphic to the 1-torus (a circle).

Furthermore, there exist, in all, 6 pairs of mutually complementary labeled simplicial 1-complexes homeomorphic to the 1-torus.





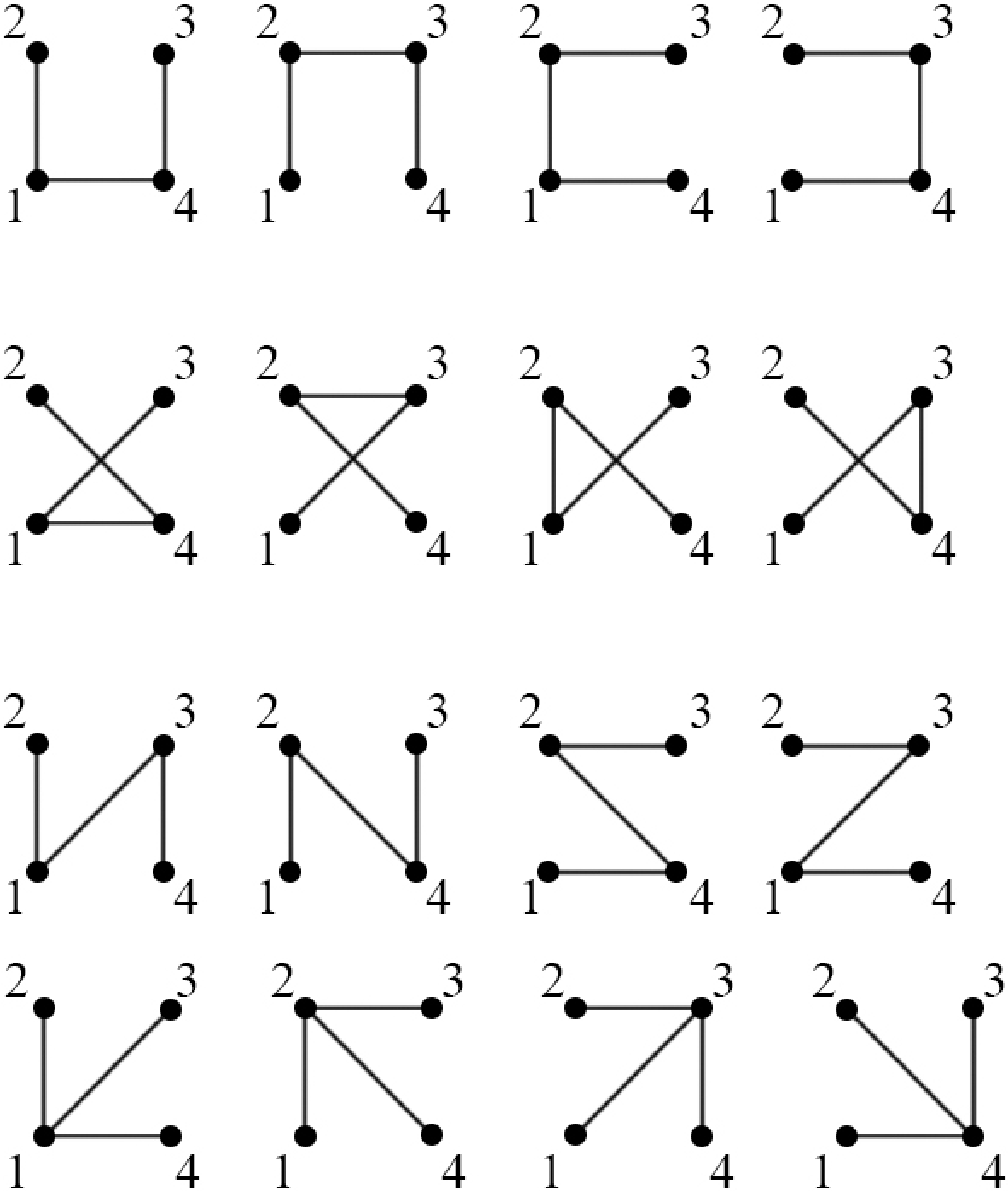
Concluding remark

The simplicial 2-torus ***T*** is a 2-dimensional analog of the simplicial 1-torus *C*_5.

Last slide

Any questions?

Appendix 1


2 3 2 3 2 3 2 3
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2 3 2 3 2 3 2 3
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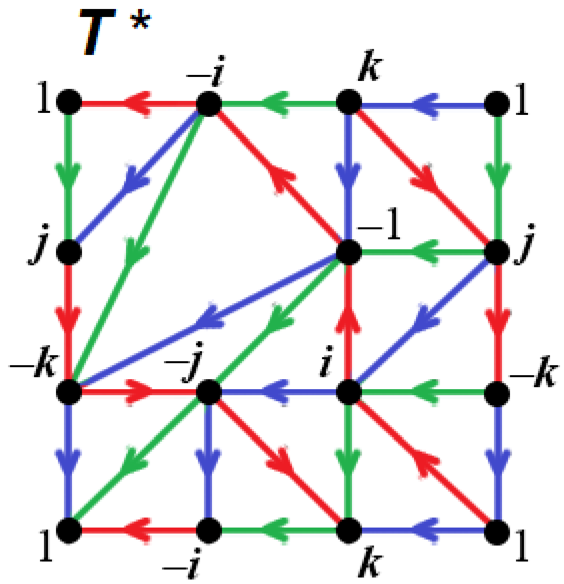
Appendix 2



Theorem

There are precisely twelve toroidal triangulations with the labeled graph ***G*** = *K*_{2, 2, 2, 2}. They are all isomorphic but pairwise different as vertex-labeled triangulations. They are presented in slides 12, 14, 15. They are obtained from the following three triangulations:

T *, (***i j***)(–***i*** –***j***)***T*** *, (***i k***)(–***i*** –***k***)***T*** *

by the action of the the quotient dihedral groups by their centers *D*_8 / *Z*(*D*_8), (respectively),

where *D*_8 stands for the automorphism groups of:

the red cycle (1, ***i***, –1, –***i***),

the green cycle (1, ***j***, –1, –***j***),

the blue cycle (1, ***k***, –1, –***k***), respectively.