

Local characteristics and tangent martingales in Banach spaces

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8ECM, June 2021

Lévy-Khinchin formula

Let $L : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ Lévy martingale.

Theorem (Lévy-Khinchin formula)

There exist unique $\sigma \geq 0$ and measure ν on \mathbb{R} s.t.

$$\mathbb{E}e^{i\theta L_t} = \exp \left\{ t \left(-\frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta x \, d\nu(x) \right) \right\}, \quad t \geq 0, \quad \theta \in \mathbb{R}.$$

Features of σ and ν

- ▶ they are unique and deterministic
- ▶ they determine distribution of L

Is there analogue for general martingales?

Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingale. Then

$$[M]_t := \mathbb{P} - \lim_{\text{mesh} \rightarrow 0} \sum_{i=1}^N |M_{t_i} - M_{t_{i-1}}|^2,$$

$0 = t_0 < \dots < t_N = t$, called **quadratic variation** of M

Example

- ▶ Let W standard Brownian motion. Then $[W]_t = t$ a.s.
- ▶ Let N standard Poisson process, $\tilde{N}_t := N_t - t$ compensated Poisson process. Then \tilde{N} martingale and $[\tilde{N}]_t = N_t$

Remark

Both M and $[M]$ have càdlàg versions (i.e. right-continuous with left limits)

Meyer-Yoeurp decomposition

Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingale. Then

- ▶ M called **continuous** if $[M]$ continuous (\Leftrightarrow path of M continuous)
- ▶ M called **purely discontinuous** if $[M]$ pure jump, i.e.
$$[M]_t = \sum_{0 \leq s \leq t} \Delta[M]_s$$

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Theorem (Meyer 1976, Yoeurp 1976)

Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingale. Then $\exists!$ local martingales M^c and M^d s.t. M^c continuous, M^d p.d., $M_0^c = 0$, and $M = M^c + M^d$

Local characteristics

Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingale. Define μ^M on $\mathbb{R}_+ \times \mathbb{R}$

$$\mu^M([0, t] \times B) = \sum_{0 \leq s \leq t} \mathbf{1}_{B \setminus \{0\}}(\Delta M_s), \quad t \geq 0, \quad B \subset \mathbb{R} \text{ Borel}$$

Let ν^M random measure on $\mathbb{R}_+ \times \mathbb{R}$ **compensator** of μ^M , i.e. unique predictable r.m. such that for any predictable $F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ with $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} F d\mu^M < \infty$

$$t \mapsto \int_{[0, t] \times \mathbb{R}} F d(\mu^M - \nu^M) \text{ local martingale}$$

Definition

Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingale, $M = M^c + M^d$ Meyer-Yoeurp decomposition. Then $([M^c], \nu^M)$ called **local characteristics** of M

$[M^c]$ cares about M^c
 ν^M cares about M^d

Lévy processes

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$([L^c], \nu^L) = (\sigma^2 t, \lambda_{\mathbb{R}_+} \otimes \nu)$ local characteristics of L

Features of σ and ν

- ▶ they are unique and deterministic
- ▶ they determine distribution of L

Properties of local characteristics

- ▶ they are unique and **predictable, but not deterministic**
- ▶ they **do not** determine distribution of martingale

Definition

Let $M, N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ martingales. Then M and N **tangent** if $([M^c], \nu^M) = ([N^c], \nu^N)$

Example

Let L^1, L^2 equidistributed Lévy martingales, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ elementary predictable. Then

$$t \mapsto \int_0^t \Phi(s) dL_s^1 \quad \text{and} \quad t \mapsto \int_0^t \Phi(s) dL_s^2$$

are tangent

Let

- ▶ $(\mathcal{F}_n)_{n=1}^{\infty}$ filtration,
- ▶ $d = (d_n)_{n \geq 1}$ and $e = (e_n)_{n \geq 1}$ \mathbb{R} -valued martingale difference sequences

Definition

d and e **tangent** iff

$$\mathbb{P}(d_n | \mathcal{F}_{n-1}) = \mathbb{P}(e_n | \mathcal{F}_{n-1}), \quad n \geq 1$$

Recall that

$$\mathbb{P}(\xi | \mathcal{F})(A) := \mathbb{E}(\mathbf{1}_A(\xi) | \mathcal{F}), \quad A \subset \mathbb{R} \text{ Borel}$$

Example

Let $(\xi_n)_{n \geq 1}$ independent mean-zero, $(v_n)_{n \geq 1}$ predictable sequence, and let $(\xi'_n)_{n \geq 1}$ independent copy of $(\xi_n)_{n \geq 1}$. Then $(v_n \xi_n)$ and $(v_n \xi'_n)$ tangent

Theorem (Zinn 1985, Hitczenko 1988)

Let d and e tangent. Then

$$\mathbb{E} \sup_{N \geq 1} \left| \sum_{n=1}^N d_n \right|^p \sim_p \mathbb{E} \sup_{N \geq 1} \left| \sum_{n=1}^N e_n \right|^p, \quad 1 \leq p < \infty$$

Theorem (Kallenberg 2017, Kwapien-Woyczyński 1991)

Let $M, N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be tangent martingales. Then

$$\mathbb{E} \sup_{0 \leq t \leq \infty} |M_t|^p \sim_p \mathbb{E} \sup_{0 \leq t \leq \infty} |N_t|^p, \quad 1 \leq p < \infty$$

What happens in Banach spaces?

UMD Banach space

Let X Banach space.

X called **UMD** if for any $p \in [1, \infty)$, X -valued m.d.s. $(d_n)_{n \geq 1}$, and for any $(\varepsilon_n)_{n \geq 1} \in \{-1, 1\}$

$$\mathbb{E} \sup_{N \geq 1} \left\| \sum_{n=1}^N \varepsilon_n d_n \right\|^p \sim_{p, X} \mathbb{E} \sup_{N \geq 1} \left\| \sum_{n=1}^N d_n \right\|^p, \quad 1 \leq p < \infty$$

Example

- ▶ Hilbert spaces,
- ▶ L^q , $q \in (1, \infty)$,
- ▶ reflexive Sobolev, Besov, Schatten class, Orlicz, variable $L^{p(\cdot)}$
- ▶ non reflexive are not UMD: L^1 , L^∞ , $C(K)$

Proposition (Burkholder '83, Bourgain '83)

X UMD iff Hilbert transform is bdd on $L^p(\mathbb{R}; X)$ for any (some)
 $1 < p < \infty$

Martingales in Banach spaces are well-studied, see e.g. *Analysis in Banach spaces* by Hytönen, van Neerven, Veraar, Weis

Let $M : \mathbb{R}_+ \times \Omega \rightarrow X$ martingale

- ▶ $\langle M, x^* \rangle$ martingale for any $x^* \in X^*$
- ▶ M **continuous** if $\langle M, x^* \rangle$ continuous for any $x^* \in X^*$
- ▶ M **purely discontinuous** if $\langle M, x^* \rangle$ p.d. for any $x^* \in X^*$

Proposition (Y. 2017)

Let X Banach space. Then X UMD iff for any martingale

$M : \mathbb{R}_+ \times \Omega \rightarrow X$ there \exists local martingales $M^c, M^d : \mathbb{R}_+ \times \Omega \rightarrow X$ such that M^c continuous, M^d p.d., $M_0^c = 0$, and $M = M^c + M^d$.

Moreover, such decomposition unique.

Let X Banach space. Denote by $Q(X^*)$ linear space of **bounded quadratic forms** on X^*

Proposition (Y. 2018)

Let X UMD Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ martingale. Then there exists $[[M]] : \mathbb{R}_+ \times \Omega \rightarrow Q(X^*)$ s.t.

$[[M]](x^*)$ and $[\langle M, x^* \rangle]$ indistinguishable $\forall x^* \in X^*$

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Let ν^M the compensator of μ^M , i.e. unique predictable r.m. such that for any predictable $F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ with $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} F d\mu^M < \infty$

$$t \mapsto \int_{[0, t] \times \mathbb{R}} F d(\mu^M - \nu^M) \text{ local martingale}$$

Definition

Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ martingale. Then $([M^c], \nu^M)$ called **local characteristics** of M

Definition

Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ martingales. Then M and N are **tangent** if they have the same local characteristics, i.e. if $([M^c], \nu^M) = ([N^c], \nu^N)$

Let $M : \mathbb{R}_+ \times \Omega \rightarrow X$ martingale. Define μ^M on $\mathbb{R}_+ \times X$

$$\mu^M([0, t] \times B) = \sum_{0 \leq s \leq t} \mathbf{1}_{B \setminus \{0\}}(\Delta M_s), \quad t \geq 0, \quad B \subset X \text{ Borel}$$

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Theorem (McConnell 1989, Hitczenko unpubl)

Let X Banach space, $1 \leq p < \infty$. Then X UMD iff for any X -valued tangent m.d.s. $(d_n)_{n \geq 1}$ and $(e_n)_{n \geq 1}$

$$\mathbb{E} \sup_{N \geq 1} \left\| \sum_{n=1}^N d_n \right\|^p \sim_{p,X} \mathbb{E} \sup_{N \geq 1} \left\| \sum_{n=1}^N e_n \right\|^p$$

Example

Let X UMD, L^1, L^2 equidistributed Lévy martingales, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow X$ elementary predictable. Then for any $1 \leq p < \infty$

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \Phi(s) dL_s^1 \right\|^p \sim_{p,X} \mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \Phi(s) dL_s^2 \right\|^p$$

Theorem (Y. 2019)

Let X UMD, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ tangent martingales. Then $\forall 1 \leq p < \infty$

$$\mathbb{E} \sup_{0 \leq t \leq \infty} \|M_t\|^p \sim_{p,X} \mathbb{E} \sup_{0 \leq t \leq \infty} \|N_t\|^p$$

Proof

- ▶ the canonical decomposition (Y. 2017)
- ▶ time-change
- ▶ continuous case: Wiener decoupling (McConnell 1989)
- ▶ discrete case (McConnell 1989, Hitczenko unpubl)
- ▶ Poisson case: random measure theory (Jacod, Shiryaev 1987, Kallenberg 2017)
- ▶ BDG inequalities in Banach spaces (Y. 2018)

Applications

- ▶ ϕ -inequalities
- ▶ decoupled tangent martingales, Jacod-Kwapień-Woyczyński approach
- ▶ stochastic integral inequalities
- ▶ Lévy-Khinchin-type formula for martingales in Banach spaces
- ▶ characteristic subordination and characteristic domination
- ▶ etc.

Many thanks!