# $L^p$ estimates for certain wave equations with Lipschitz coefficients

Pierre Portal, Australian National University
joint work with D. Frey (Karlsruhe)

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**Recorded talk:**

https://www.youtube.com/watch?v=BVMXb2bFRQc

# THE RESULT

For $j \in \{1, \ldots, 2d\}$, let $a_j \in C^{0,1}(\mathbb{R})$ with $\frac{d}{dx}a_j \in L^\infty$, and assume that there exist $0 < \lambda \leq \Lambda$ such that $\lambda \leq a_j(x) \leq \Lambda$ for all $x \in \mathbb{R}$. Define $\widetilde{a}_j : x \mapsto a_j(x_j)$ and, for $\xi \in \mathbb{R}^d$,

$$i\xi.D_a := \sum_{j=1}^{d} \xi_j \begin{pmatrix} 0 & -i\partial_j a_{j+d} \\ ia_j\partial_j & 0 \end{pmatrix},$$

$$L := D_a.D_a = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix},$$

where $L_1 := -\sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial j$ and $L_2 := -\sum_{j=1}^{d} a_j \partial_j a_{j+d} \partial j$.

**Theorem:** Let $p \in (1, \infty)$ and $s_p = (d-1)|\frac{1}{p} - \frac{1}{2}|$. For each $t \in \mathbb{R}$, the operator $(I + \sqrt{L})^{-s_p} \exp(it\sqrt{L})$ is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^2)$. Moreover $\exp(it\sqrt{L})$ leaves a scale of Hardy-Sobolev spaces $H^{p,s}_{FIO,a}$ invariant.

# Wave packet transform

**Definition:** For $\sigma > 0$, $W_\sigma \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d \times S^{d-1}))$ is defined by

$$W_\sigma(f)(x,\omega) := \Psi(\sigma\omega.\nabla)\psi(\sigma^{\frac{1}{2}}\omega_1.\nabla)...\psi(\sigma^{\frac{1}{2}}\omega_{d-1}.\nabla)f(x),$$

where $(\omega, \omega_1, ..., \omega_{d-1})$ is an orthonormal basis, $\Psi, \psi \in C_c^\infty$ and $supp\Psi \subset [\frac{1}{4}, 2]$.

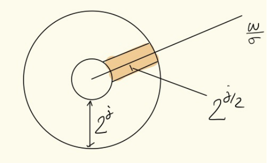


Figure: $|\eta| \sim 2^j$ and $|\omega - \frac{\eta}{|\eta|}| \lesssim 2^{-\frac{j}{2}}$.

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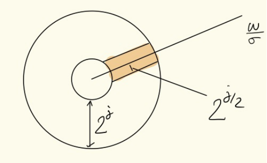


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**Energy packets/dyadic-parabolic decomposition:**

$$\|f\|^2_{L^2(\mathbb{R}^d)} \sim \int_0^1 \|W_\sigma f\|^2_{L^2(\mathbb{R}^d \times S^{d-1})} \frac{d\sigma}{\sigma} + \int_1^\infty \|\Psi(\sigma^2\Delta)f\|^2_{L^2(\mathbb{R}^d)} \frac{d\sigma}{\sigma}$$

# Transport group

**Transport:** For $\xi \in \mathbb{R}^d$, $\exp(i\xi \cdot D_a)$ defines a $d$-parameter bounded $C_0$-group of operators on $L^p$, for all $p \in (1, \infty)$. It has finite speed of propagation.

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**Transference (Coifman-Weiss):**
For $X = L^p(\mathbb{R}^d)$, $q \in (1, \infty)$, and all $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$\|\int_{\mathbb{R}^d} \widehat{\psi}(\xi) \exp(i\xi \cdot D_a) f d\xi\|_X \lesssim \|T_\psi \otimes I_X\|_{B(L^q(\mathbb{R}^d;X))} \|f\|_X.$$

# Adapted Hardy-Sobolev spaces

**Square function spaces over phase space:**

$$\|F\|_{L^p(T^{p,2})} := \left( \int\limits_{\mathbb{R}^d \times S^{d-1}} \left( \int_0^\infty \fint_{B(x,\sigma)} |F(y,\omega,\sigma)|^2 \, dy \frac{d\sigma}{\sigma} \right)^{p/2} dx d\omega \right)^{\frac{1}{p}}.$$

# ADAPTED HARDY-SOBOLEV SPACES

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**Classical Hardy space $H^1$:**

$$\|f\|_{H^1(\mathbb{R}^d)} \sim \|(x,\omega,\sigma) \mapsto \Psi(\sigma\nabla)f(x)\|_{L^1(T^{1,2}(\mathbb{R}^d))}.$$

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**Hardy-Sobolev FIO spaces associated with $D_a$:**

$$\|f\|_{H^{p,s}_{FIO,a}(\mathbb{R}^d)} := \|\omega \mapsto [(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\psi_{\omega,\sigma}(D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.$$

# DECAY ESTIMATES

For every $M \in \mathbb{N}$, there exists $C_M > 0$ such that for all $E, F \subset \mathbb{R}^d$ Borel sets, $\sigma \in (0,1)$ and $\omega \in S^{d-1}$, we have

$$\|1_E \psi_{\omega,\sigma}(D_a)(1_F f)\|_{L^2(\mathbb{R}^d)} \leq C_M \sigma^{-\frac{d}{2}} (1 + \frac{d_\omega(E,F)}{\sigma})^{-M} \|1_F f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$, where

$$d_\omega(x,y) := |\langle \omega, x-y \rangle| + \sum_{j=1}^{d-1} \langle \omega_j, x-y \rangle^2 \quad \forall x, y \in \mathbb{R}^d.$$