Asymptotic consensus in the Hegselmann-Krause model with finite speed of information propagation

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The Hegselmann-Krause model (2002)

- Prototypical model for opinion dynamics: agents adapt their opinions to others', with confidence depending on the difference in opinions.
- For $i = 1, \dots, N$, $x_i = x_i(t) \in \mathbb{R}^d$ subject to

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(x_j - x_i)$$

The influence function $\psi \geq 0$ bounded, typically nonincreasing.

- For $\psi > 0$: convergence to global consensus

$$\lim_{t \to \infty} x_i = \bar{x} \qquad \text{for all } i = 1, \dots, N,$$

with

$$\bar{x} = \frac{1}{N} \sum_{i=0}^{N} x_i(0)$$

Finite propagation speed

- Speed of light $\mathfrak{c} > 0$
- Agent located at $x_i = x_i(t)$ at time $t > 0$ observes the position of the agent x_j at time $t - \tau_{ij}$, where τ_{ij} solves

$$\mathfrak{c}\tau_{ij}(t) = |x_i(t) - x_j(t - \tau_{ij}(t))|$$

- Unique solvability guaranteed iff

$$|\dot{x}_j(t)| \le \mathfrak{s} \qquad \text{for all } t \in \mathbb{R}$$

with

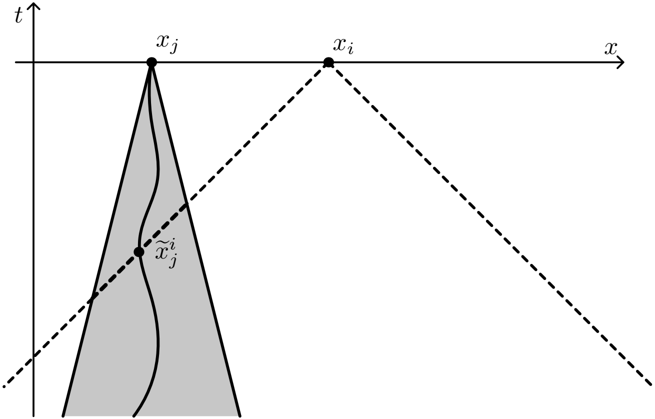
$$\mathfrak{s} < \mathfrak{c}$$

- Introduce the notation

$$\tilde{x}_j^{\,i} := x_j(t - \tau_{ij}(t))$$

... information about position of j received by i at time t.

Finite propagation speed



Finite propagation speed

- We study the system

$$\dot{x}_i = \frac{1}{N-1}\sum_{j=1}^{N} \psi(|\tilde{x}_j^{\,i} - x_i|)\,(\tilde{x}_j^{\,i} - x_i)$$

with $\tilde{x}_j^{\,i} := x_j(t - \tau_{ij}(t))$ unique solution of

$$\mathfrak{c}\tau_{ij}(t) = |x_i(t) - x_j(t - \tau_{ij}(t))|$$

- Subject to the $\mathfrak{s}$-Lipschitz continuous initial datum

$$x_i(t) = x_i^0(t) \qquad \text{for } i = 1, \dots, N, \quad t \le 0$$

- Central speed limit assumption:

$$\mathfrak{s} := \sup_{r>0} \psi(r) r < \mathfrak{c}$$

then $|\dot{x}_i| \le \mathfrak{s}$ for all $i = 1, \dots, N$.

Results I: Well posedness

- Local existence and uniqueness of solutions: Based on an adaptation of Picard-Lindelöf theorm - contraction in the space of $\mathfrak{s}$-Lipschitz continuous functions.
- Initial datum

$$x^0 \in C_{\mathfrak{s}}([-S^0, 0]; \mathbb{R}^d)^N$$

with

$$S^0 := \frac{d_x(0)}{\mathfrak{s} - \mathfrak{c}}, \qquad d_x(t) := \max_{i,j \in \{1, \cdots, N\}} |x_i(t) - x_j(t)|$$

- Global due to

$$|\dot{x}_i| \leq \sup_{r>0} \psi(r) r = \mathfrak{s} < \mathfrak{c}$$

Results II: Consensus in 1D

- Theorem. In 1D, let $\mathfrak{s} < \mathfrak{c}$ and $\psi > 0$. Then

$$\lim_{t\to\infty} d_x(t) = 0$$

exponentially with explicit rate; but no conservation of mean.

- Proof based on preservation of ordering

$$x_1(t) \le x_2(t) \le \cdots \le x_N(t)$$

so that $d_x = x_N - x_1$,

- and on the monotonicity property

$$x_i < x_j \quad \Rightarrow \quad x_i < \tilde{x}_j^{\,i}, \qquad \tilde{x}_i^{\,j} < x_j, \qquad \tilde{x}_i^{\,j} < \tilde{x}_j^{\,i}$$

so that

$$\dot{x}_1 \ge 0, \quad \dot{x}_N \le 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} d_x(t) \le 0.$$

Results III: Consensus in multi-D

- Theorem. Let $\mathfrak{s} < \mathfrak{c}$. Then
$$\frac{\mathrm{d}}{\mathrm{d}t} d_x \le \left(\frac{2\mathfrak{s}}{\mathfrak{c}-\mathfrak{s}} \overline{\psi} - \underline{\psi} \right) d_x$$
  with
$$\underline{\psi} := \min_{r\in[0,d_x(0)]} \psi(r), \qquad \overline{\psi} := \max_{r\in[0,d_x(0)]} \psi(r).$$
- Consequently, exponential convergence to consensus whenever
$$\frac{2\mathfrak{s}}{\mathfrak{c}-\mathfrak{s}} < \underline{\psi}/\overline{\psi}$$
  In particular, $3\mathfrak{s} < \mathfrak{c}$ is necessary.
- Proof based on triangle and Cauchy-Schwarz inequalities,
$$|\widetilde{x}_j{}^i - x_j| \le \frac{\mathfrak{s}}{\mathfrak{c}-\mathfrak{s}} d_x(t)$$

Which mean-field limit?

Collaboration with O. Tse (Eindhoven)

Mean-field limit

- Remark. For the system with a fixed delay $\tau > 0$ the mean-field limit is given by the Fokker-Planck equation

$$\partial_t f_t + \nabla \cdot (F[f_{t-\tau}] f_t) = 0$$

for $f \in C([-\tau, T]; \mathcal{P}(\mathbb{R}^d))$, with

$$F[f_{t-\tau}](x) = \int_{\mathbb{R}^d} \psi(|x-y|)(x-y)\,\mathrm{d}f_{t-\tau}(y)$$

- "Intuitive guess" (WRONG!) for our system:

$$\partial_t f_t + \nabla_x \cdot (G_t[f] f_t) = 0,$$

with

$$G_t[f](x) = \int_{\mathbb{R}^d} \psi(|x-y|)(x-y) f(t - \mathfrak{c}^{-1}|x-y|, y)\,\mathrm{d}y$$

Mean-field limit

- Description in terms of $\varrho \in \mathcal{P}(\Omega_{\mathfrak{s}})$ with

$$\Omega_{\mathfrak{s}} := \{\mathfrak{s} - \text{Lipschitz continuous functions on } (-\infty, T]\}$$

- Denote $K(z) := \psi(|z|)z$.
- Study the object $x \in \Omega_{\mathfrak{s}}$ such that

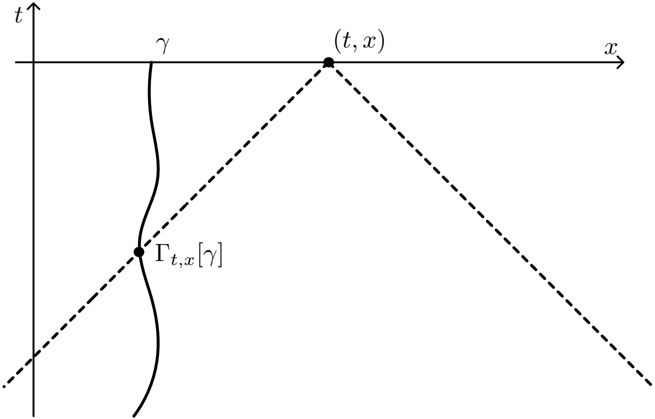
$$\dot{x}(t) = \int_{\Omega_{\mathfrak{s}}} K(\Gamma_{t,x(t)}[\gamma] - x(t))\,\mathrm{d}\varrho(\gamma)$$

with $\Gamma_{t,x} : \Omega_{\mathfrak{s}} \mapsto \mathbb{R}^d$ defined as the unique solution $y \in \mathbb{R}^d$ of

$$y = \gamma(t - \mathfrak{c}^{-1}|x - y|)$$

- Using the Banach contraction theorem, construct $\varrho = \text{law}(x)$.

Definition of $\Gamma_{t,x}$



Fokker-Planck equation?

- Goal: Pass from $\varrho \in \mathcal{P}(\Omega_\mathfrak{s})$ to $f \in C([-\infty, T]; \mathcal{P}(\mathbb{R}^d))$.
- Define f_t as the time-slice $f_t := T_t \# \varrho \in \mathcal{P}(\mathbb{R}^d)$ with $T_t : \gamma \mapsto \gamma(t)$.
- Question: Given a solution $\varrho = \mathrm{law}(x)$, can we write a closed equation for $f_t := T_t \# \varrho$?
- Answer: Yes, if we were able to express

$$G_t[\varrho](x) := \int_{\Omega_\mathfrak{s}} K(\Gamma_{t,x}[\gamma] - x) \, \mathrm{d}\varrho(\gamma)$$

in terms of f_t.

Fokker-Planck equation?

- Apply a co-ordinate transform $t \mapsto \tilde{t}_x$ which turns $\Gamma_{t,x}$ into $T_{\tilde{t}_x}$, i.e.,

$$\Gamma_{t,x}[\gamma] = T_{\tilde{t}_x}[\gamma] = \gamma(\tilde{t}_x)$$

... but (of course) $\tilde{t}_x$ depends on γ, i.e., $\gamma(\tilde{t}_x[\gamma])$

- One may integrate in time, which gives (drop x for simplicity)

$$\int_{\Omega_{\mathfrak{s}}} \left(\int_{-\infty}^{\infty} \psi(t, \Gamma_t(\gamma))\, \mathrm{d}t \right) \mathrm{d}\varrho(\gamma)$$

$$= \int_{\Omega_{\mathfrak{s}}} \int_{-\infty}^{\infty} \psi(\tilde{t} + \mathfrak{c}^{-1}|\gamma(\tilde{t})|, \gamma(\tilde{t})) \left(\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}[\gamma] \right) \mathrm{d}\tilde{t}\, \mathrm{d}\varrho(\gamma),$$

with a test function $\psi = \psi(t, y)$ and

$$\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}[\gamma] = 1 + \mathfrak{c}^{-1} \frac{\gamma(\tilde{t}) \cdot \dot{\gamma}(\tilde{t})}{|\gamma(\tilde{t})|}$$

- So the answer is: No (classical) Fokker-Planck equation.

Thank you for your attention!