The L_p dual Minkowski problem and polytopal approximation

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Minkowski problem I - Gauss curvature

- K is a convex body in $\mathbb{R}^n$ with ∂K is C_+^2
- $\nu_K(x)$ is exterior unit normal at $x \in \partial K$
- $f_K(\nu_K(x)) = 1/\kappa_K(x) > 0$ is curvature function where $\kappa_K(x)$ is the Gauss curvature at $x \in \partial K$

Observation (Minkowski)

$$\int_{S^{n-1}} u \cdot f_K(u)\, du = o. \tag{1}$$

Minkowski problem (E.g. Inverse problem of short wave diffraction)
For continuous $f : S^{n-1} \to \mathbb{R}_+$ satisfying (1), find K with ∂K is C_+^2 such that $f(\nu_K(x)) = 1/\kappa(x)$ for $x \in X$.
Monge-Ampere type differential equation on S^{n-1}:

$$\det(\nabla^2 h + h\, I) = f$$

where $h(u) = h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ support function.

Minkowski problem II - Surface area measure

$\partial' K$ - smooth boundary points
$\nu_K(x)$ - unique exterior normal at $x \in \partial' K$
S_K - surface area measure on S^{n-1} of a convex body K in $\mathbb{R}^n$

$$S_K(\omega) = \mathcal{H}^{n-1}\left\{x \in \partial' K : \nu_K(x) \in \omega\right\}$$

for Borel $\omega \subset S^{n-1}$ where $\mathcal{H}^{n-1}(\cdot)$ $(n-1)$-Hausdorff measure

- ∂K is $C^2_+ \Longrightarrow dS_K = f_K \, d\mathcal{H}^{n-1}$
- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i).$$

Minkowski problem Find K with $\mu = S_K$ if $\int_{S^{n-1}} u \, d\mu(u) = o$

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$
- Uniqueness up to translation

L_0 surface area/Cone volume measure

$dV_K = h_K dS_K$ - cone volume measure or L_0 surface area measure on S^{n-1} if $o \in K$ (Firey (1974), Gromov, Milman (1986))

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$V_K(\{u_i\}) = h_K(u_i)\mathcal{H}^{n-1}(F_i) = n \cdot V(\mathrm{conv}\{o, F_i\}).$$

- ∂K is $C^2_+ \Longrightarrow dV_K = h_K f_K \, d\mathcal{H}^{n-1}$
- $V_K(S^{n-1}) = nV(K)$.

Naor, Werner, Paouris, Stancu... used it say for L_p balls

Logarithmic Minkowski problem

Monge-Ampere type differential equation on S^{n-1} for $h = h_K$ if μ has a density function f:

$$h \det(\nabla^2 h + h\, I) = f$$

Theorem (Chen, Li, Zhu 2017)

$\mu = V_K$ for some convex body $o \in K$ if
$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot V(K)$ for any linear subspace $L \neq \{o\}, \mathbb{R}^n$

Remark The condition does not charactherize V_K, there are some other conditions - CHARACTERIZATION FULL OPEN, not even a conjecture is known

L_p surface area measures

L_p surface area measures (Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K = h_K^{-p}\, dV_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0} = V_K$
- $S_{K,-n}$ - $\mathrm{SL}(n)$ invariant curvature function $f_K h_K^{n+1}$

Theorem (Chou&Wang, Chen&Li&Zhu)

If $p > 0$, $p \neq 1, n$, then any finite Borel measure μ on S^{n-1} not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Remark Possibly $o \in \partial K$ with ∂K is $C^{1,\alpha}$ if $0 < p < n$

Ideas

- Minimize $\int_{S^{n-1}} h_C^p \, d\mu$ under the condition $V(C) = 1$
- Weak approximation by "nice" measures

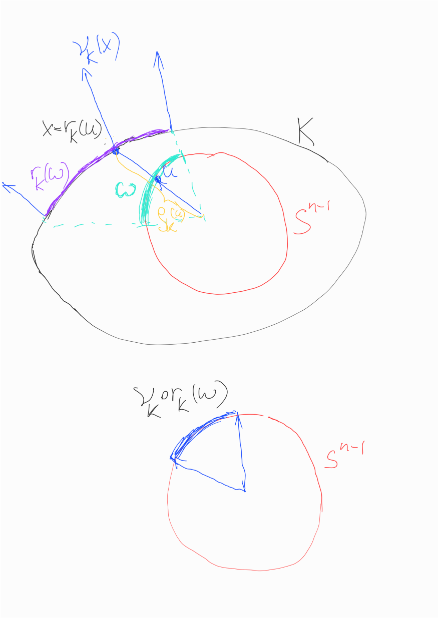

Figure: Integral Curvature Measure

Alexandrov's problem

$o \in \mathrm{int} K,\ u \in S^{n-1} \Longrightarrow r_K(u) = \varrho_K(u) \cdot u \in \partial K$

$\varrho_K(u)$=radial function

Alexandrov's Integral Gauss curvature, 1940

$\omega \subset S^{n-1}$ Borel $\Longrightarrow C_K(\omega) = \mathcal{H}^{n-1}(\nu_K \circ r_K(\omega))$

Theorem (Alexandrov)

For Borel measure μ on S^{n-1} with $\mu(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1})$
$\mu = C_K$ if and only if for any $\omega \subset S^{n-1}$, $\omega \neq S^{n-1}$, we have

$$\mu(\omega) < \mathcal{H}^{n-1}(\{x \in S^{n-1} : \exists y \in \omega,\ \langle x, y\rangle > 0\}).$$

Dual curvature measures for $q \in \mathbb{R}$

Huang, Lutwak, Yang, Zhang

$$\widetilde{C}_{K,q}(\nu_K \circ r_K(\omega)) = \int_\omega \varrho_K^q(u)\, du \quad \text{for } \omega \subset S^{n-1}$$

$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q(u)\, du$ dual intrinsic volume ($\widetilde{V}_n(K) = V(K)$)

Examples

- $\widetilde{C}_{K,0} = \alpha C_{K^*}$
- $\widetilde{C}_{K,n} = V_K$

L_p dual curvature measures for $q \in \mathbb{R}$

$$d\widetilde{C}_{K,p,q} = h_K^{-p}\, d\widetilde{C}_{K,q}$$

Examples

- $\widetilde{C}_{K,0,q} = \widetilde{C}_{K,q}$
- $\widetilde{C}_{K,p,n} = S_{K,p}$

L_p dual Minkowski problem

L_p dual Minkowski problem: find K with $\widetilde{C}_{K,p,q} = \mu$

- $p > 1$ and $q > 0$ (B. & Ferenc Fodor)
- $p \geq 0$ and $q < 0$ (Huang&Zhao, Chen&Wang&Li)
- $p > q$ and μ has C^α density (Huang&Zhao)

Idea Minimize $\int_{S^{n-1}} h_C^p \, d\mu$ assuming $\widetilde{V}_q(C) = 1$

Remark Possible $o \in \partial K$ needs to be allowed even if μ has positive C^α density (say when $1 < p < q$) - L_p dual Minkowski problem is more carefully stated in that case

Discrete L_p dual Minkowski problem - polytopes

$p > 1$ and $q > 0$

Polytopes with given exterior normals $u_1, \dots, u_k \in S^{n-1}$

$$\forall v \in S^{n-1} \ \exists u_i \ \langle v, u_i \rangle > 0$$

For $h_1, \dots, h_k > 0$

$$Q(h_1, \dots, h_k) = \{\langle x, u_i \rangle \le h_i, \ i = 1, \dots, k\}$$

$$\mathcal{P} = \left\{ Q(h_1, \dots, h_k) : \widetilde{V}_q(Q(h_1, \dots, h_k)) = 1 \right\}$$

Given measure μ on $\{u_1, \dots, u_k\}$, $\mu(u_i) > 0$, find

$$\min \left\{ \sum_{i=1}^{k} h_i^p \mu(u_i) : Q(h_1, \dots, h_k) \in \mathcal{P} \right\}$$

$\exists$ optimal $Q_0 \in \mathcal{P}$, $o \in \operatorname{int} Q_0 \Longrightarrow \exists \ \lambda > 0$, $\mu = \widetilde{C}_{\lambda Q_0, p, q}$

Remark $o \in \operatorname{int} Q_0$ follows from μ discrete

General μ - approximation by polytopes

μ finite Borel measure on S^{n-1} satisfying

$$\forall v \in S^{n-1} \; \mu\left(\{u \in S^{n-1} : \langle v, u\rangle > 0\}\right) > 0 \tag{2}$$

μ weakly approximated by discrete $\mu_m \implies \mu_m = \widetilde{C}_{Q_m,p,q}$

(2) $\implies \{Q_m\}$ bounded $\implies \exists\; Q_{m'} \to K,\; \mu = \widetilde{C}_{K,p,q}$ $\square$

Uniqueness of the solution of dual L_p-Minkowski problem

- $p > q$ - Unique solution (Dongmeng Xi and Zhen Zhang)
- $p < 0$ and $q > 1$ - Non-uniqueness, even if K is assumed to be rotationally symmetric, o-symmetric with C^∞_+ boundary (Qi-rui Li, Jiankun Liu, Jian Lu)