Closures of solvable permutation groups

(based on the paper with E. A. O'Brien, I. Ponomarenko, and E. Vdovin)

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Ω is a finite set, $G \leq \mathrm{Sym}(\Omega)$, and m is a positive integer

G acts componentwisely on Ω^m: $(\alpha_1, \ldots, \alpha_m)^g = (\alpha_1^g, \ldots, \alpha_m^g)$

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The *m-closure* $G^{(m)}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ m-equivalent to G.

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$$G^{(m)} = \{g \in \mathrm{Sym}(\Omega) : \Delta^g = \Delta, \Delta \in \mathrm{Orb}_m(G)\} = \mathrm{Aut}(\mathrm{Orb}_m(G)).$$

$G^{(m)}$ is the full automorphism group of specific (induced by group) combinatorial structure on Ω consisting of (colored) m-ary relations.

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The m-closure of m-transitive group $G \leq \mathrm{Sym}(\Omega)$ is $\mathrm{Sym}(\Omega)$, so if $\Omega = \underbrace{\Delta_1 \cup \ldots \cup \Delta_s}_{1-\text{orbits}}$, then $G^{(1)} = \mathrm{Sym}(\Delta_1) \times \ldots \times \mathrm{Sym}(\Delta_s)$.

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H. Wielandt (1969):

Suppose $m \geq 2$. Then

1. G is abelian $\Rightarrow$ $G^{(m)}$ is abelian
2. G is a p-group $\Rightarrow$ $G^{(m)}$ is a p-group
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D. Churikov, I. Ponomarenko (2020):

G is nilpotent $\Rightarrow$ $G^{(m)}$ is nilpotent.

Primitive case

M. Liebeck, C. Praeger, J. Saxl 1988; and C. P and J. S, 1992:

If $m \geq 2$ and $G \leq \mathrm{Sym}(\Omega)$ is primitive, then either $\mathrm{Soc}(G) = \mathrm{Soc}(G^{(m)})$, or one of the following holds:

1. G is m-transitive for $2 \leq m \leq 5$;
2. $m = 3$, $|\Omega| = 15$, and $A_7 \simeq G < G^{(3)} \simeq A_8$;
3. $m = 2$ and G and $G^{(2)}$ are known almost simple groups;
4. G and $G^{(m)}$ preserve a product decomposition $\Omega = \Delta^k$, $k \geq 2$, and G^{Δ} and $(G^{(m)})^{\Delta}$ are groups from (1)–(3).

In particular, if $m \geq 6$, then $\mathrm{Soc}(G) = \mathrm{Soc}(G^{(m)})$.

The socle $\mathrm{Soc}(G)$ of G is the subgroup of G generated by all its minimal normal subgroups.

Solvable Permutation Groups

There are 2-transitive solvable groups, say, $\mathrm{AGL}(1,p)^{(2)} = \mathrm{Sym}(p)$ for a prime p, so assuming $p \geq 5$, we get

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Main Result

E. O'Brien, I. Ponomarenko, A. V., and E. Vdovin (2020):

If $m \geq 3$ and G is solvable, then $G^{(m)}$ is solvable.

Basic facts on m-closures

Below $G, H \leq \mathrm{Sym}(\Omega)$.

Lemma 1

$G \leq G^{(m)}$, $G^{(m)} = (G^{(m)})^{(m)}$, and $G \leq H$ implies $G^{(m)} \leq H^{(m)}$.

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Lemma 2

Suppose that $m \geq 2$. Then

1. G is m-closed, if there is an $(m-1)$-closed one-point stabilizer;
2. G is $(m+1)$-closed, if an $(m-1)$-point stabilizer has a faithful regular orbit.

Corollary

A point stabilizer of G has a faithful regular orbit $\Rightarrow$ G is 3-closed.

Closures of products of permutation groups

Let $K \leq \mathrm{Sym}(\Gamma)$ and $L \leq \mathrm{Sym}(\Delta)$.

Lemma 3 (folklore)

If $K \times L$ acts on $\Gamma \sqcup \Delta$, then $(K \times L)^{(m)} = K^{(m)} \times L^{(m)}$, $m \geq 1$.

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Lemma 4 (L. A. Kalužnin, M. Klin, 1976)

If $K \wr L$ acts on $\bigsqcup_{\delta \in \Delta} \Gamma_\delta$, then $(K \wr L)^{(m)} = K^{(m)} \wr L^{(m)}$, $m \geq 2$.

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$(\mathrm{Sym}(2) \wr \mathrm{Alt}(3))^{(2)} = \mathrm{Sym}(2) \wr \mathrm{Sym}(3) \not\leq \mathrm{Sym}(2) \wr \mathrm{Alt}(3)$.

S. Evdokimov, I. Ponomarenko (2001):

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Lemma 5 (New)

If $K \wr L$ acts on Γ^{Δ} primitively, then $(K \wr L)^{(3)} \leq K^{(3)} \wr L^{(3)}$.

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Since $G^{(m)} \leq G^{(3)}$ for $m \geq 3$, it suffices to prove

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Let G be a counterexample of the least possible degree.

Claim 1

G is basic, i. e. G is primitive and does not preserve any product decomposition of Ω.

Hint: Apply Lemmas 3–5.

Since G is a primitive solvable group, G is affine, that is Ω can be identified with a vector space of size p^d,

$$G \leq \mathrm{AGL}(d, p) \qquad H \leq \mathrm{GL}(d, p),$$

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Claim 2

G is neither subgroup of $\mathrm{A\Gamma L}(1, p^d)$, nor 2-transitive.

- $\Gamma\mathrm{L}(1, p^d)$ is 2-closed (J. Xu et al., 2011) + Lemma 2.
- By Huppert's classification of solvable 2-transitive groups, if $G \nleq \mathrm{A\Gamma L}(1, p^d)$, then $p^d \in \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$.

An irreducible group $H \leq \mathrm{GL}(V)$ is imprimitive (as a linear group) if there is a subspace $U \subset V$ such that V is a direct sum of U^h, $h \in H$, and primitive otherwise.

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Since the m-closure operator preserves the inclusion, we may assume that a point stabilizer H of G is a maximal solvable primitive subgroup of $GL(d, p)$.

Suprunenko's theory (1972) shows that any such group H is characterized (in some precise sense) by four integers, which we refer to as parameters of G.

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- Claim 2 $\Rightarrow$ G is neither subgroup of $\mathrm{A\Gamma L}(1, p^d)$, nor 2-trans.
- H has a faithful regular orbit $\Rightarrow$ G is 3-closed by Lemma 2.
- Otherwise there are only 102 sets of parameters of G (Al. Vasil'ev (not me!), E. Vdovin, Y. Yang, 2020).

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- Claim 2 $\Rightarrow$ G is neither subgroup of $\mathrm{A\Gamma L}(1, p^d)$, nor 2-trans.
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- Otherwise there are only 102 sets of parameters of G (Al. Vasil'ev (not me!), E. Vdovin, Y. Yang, 2020).

In order to complete the proof of the main theorem, we check with the help of computer computations that G is 3-closed for the remaining 102 sets of parameters.

Tools: GAP packages IRREDSOL and COCO2, and for some large cases additional computations in MAGMA.

Theorem (E. A. O'Brien, I. Ponomarenko, A. V., and E. Vdovin)

If $m \geq 3$ and G is solvable, then $G^{(m)}$ is solvable.

- E. A. O'Brien, I. Ponomarenko, A. V. Vasil'ev, and E. Vdovin, *The 3-closure of a solvable permutation group is solvable*, 2020, arXiv:2012.14166, subm. to J. Algebra.
- Y. Yang, A. S. Vasil'ev, and E. Vdovin, *Regular orbits of finite primitive solvable groups, III*, 2020, arXiv:1612.05959, subm. to J. Algebra.