G_2- and Spin(7)-structures by means of vector cross products

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Motivations and outline of my talk

• G_2 and Spin(7) are currently most exciting objects in GDbDF and M-F theory.

• There are many deep results and sophisticated techniques in complex geometry and CR-geometry.

• It is natural to continue Gray's path on searching complex and CR- structures associated to G_2 and Spin(7)-manifolds.

1. VCPs and G_2-and Spin(7)-structures.

2. Parallel VCPs and formally Kähler structures on higher dimensional loop spaces.

3. Parallel VCPs, formally integrable structure and Frörlicher-Nijenhuis bracket.

4. Calibrated submanifolds, complex submanifolds and parallel VCPs.

5. CR-twistor spaces over manifolds with G_2- and Spin(7)-structure.

1. VCPs, G_2-and Spin(7)-structure

$\chi \in \Omega^r(M, TM)$ on (M, g) is a VCP iff

$$\langle \chi(v_1, \cdots, v_r), v_i \rangle = 0 \text{ for } 1 \le i \le r,$$

$$\langle \chi(v_1, \cdots, v_r), \chi(v_1, \cdots, v_r) \rangle = \|v_1 \wedge \cdots \wedge v_r\|^2$$

For a VCP $\chi \in \Omega^r(M, TM)$ we associate the VCP-form $\varphi_\chi \in \Omega^{r+1}(M)$ as follows

$$\varphi_\chi(v_1, \cdots, v_{r+1}) = \langle \chi(v_1, \cdots, v_r), v_{r+1} \rangle_g.$$

- χ on (M^n, g) is defined uniquely by φ_χ.

- $Stab_{\mathsf{GL}(\mathbf{R}^7)}(\varphi_\chi^3) = \mathsf{G}_2 \subset \mathsf{SO}(7)$.

- $Stab_{GL(\mathbf{R}^8)}(\varphi_\chi^4) = \mathsf{Spin}(7) \subset \mathsf{SO}(8)$.

- G_2/Spin(7)-structures $\longleftrightarrow \{\, g \cdot \varphi_\chi\}$.

- VCPs in dimension 3,7, 8 can be expressed in terms of normed algebra operations.

- A $(n-1)$-fold VCP on a Riemannian manifold (M^n, g) is defined uniquely by the conformal class of g.

2. Parallel VCPs and formally Kähler structure on higher dimensional loop spaces

(2) Brylinski (1993): The space $\mathcal{L}_{i,f}(M^3)$ of unparameterized freely immersed loops on a (M^3, g) has a formally Kähler structure.

(3) LeBrun (1993): The space $B_e^+(S, M)$ of unparameterized embedded oriented submanifolds diffeomorphic to $S \subset (M, g)$ has formally Kähler structure, if codim $S = 2$ and M is oriented.

(4) Lee-Leung (2007): The space $B_e^+(S, M)$ has an almost Kähler structure, if (M, g) has a closed r-fold VCP and $\dim S = r - 1$.

(5) Verbitsky (2010): The space $\mathcal{L}_e(S^1, M^7)$ has a formally Kähler structure, if (M^7, g) is a torsion-free G_2-manifold.

(6) Fiorenza-L. (2019): The space $B_{i,f}^+(S, M)$ of unparameterized freely immersed submanifolds diffeomorphic to S in (M, g) has a formally Kähler structure, if (M, g) has a parallel r-fold VCP and $\dim S = r - 1$.

• Brylinski proof uses a trick. Lempert (1993) proved that the ACS J on $\mathcal{L}_{i,f}(M^3)$ is weakly integrable by using LeBrun's CR twistor space over a 3-manifold M^3. Using Rossi's CR-twistor space over $B_e^+(S,M)$, when codim $S = 2$, LeBrun proved the formal integrability of the ACS J. Verbitsky constructed a CR-twistor space for the proof of the formal integrability of J on $B_{i,f}^+(S^1,M^7)$. Fiorenza-L. proved the formal integrability of J on $B_{i,f}^+(S,M)$ by showing that $\nabla^{LC} J = 0$.

3. Parallel VCPs, formally integrable structures and Frörlicher-Nijenhuis bracket

- $\nabla^{LC} J = 0 \iff N_J = 0$.

- Kotaro-L.-Schwachhöfer (2018) A natural generalization of that equivalence for parallel VCP is to use Frölicher-Nijnehuis bracket on the graded Lie algebra $(\Omega^*(M, TM), [,]^{FN})$.

For $K = \alpha^k \otimes X \in \Omega^*(M, TM)$ we let

$$\imath_{\alpha^k \otimes X} \alpha^l := \alpha^k \wedge (\imath_X \alpha^l) \in \Omega^{k+l-1}(M),$$

and extend it $\mathbf{R}$-linearly on $\Omega^*(M, TM)$.

$$\mathcal{L} : \Omega^*(M, TM) \to Der(\Omega^*(M)),\ K \mapsto \mathcal{L}_K,$$

$$\mathcal{L}_K := \mathcal{L}(K) := [\imath_K, d] \in Der(\Omega^*(M)).$$

- $\mathcal{L}$ is injective and induces the Lie bracket on $\Omega^*(M, TM)$.

- For (M, g) we define the contraction

$$\Lambda^k V^* \longrightarrow \Lambda^{k-1} V^* \otimes V, \; \varphi \mapsto \hat{\varphi} := (\imath_{e_i}\varphi) \otimes (e^i)^{\#},$$

Theorem (LKS, 2018) 1. Let φ be a parallel differential form of even degree on (M, g). Then $[\hat{\varphi}, \hat{\varphi}]^{FN} = 0$.

2. Let φ be a differential 4-form with $Stab(\varphi) \subset G_2$ on a manifold M^7(resp. $Stab(\varphi) \subset Spin(7)$ on a manifold M^8). Then $[\hat{\varphi}, \hat{\varphi}]^{FN} = 0$ iff φ is parallel.

• The identity $[\widehat{\varphi}, \widehat{\varphi}]^{FN} = 0$ led us to study almost formality of G_2 and Spin(7)-manifolds, which I shall not discuss here. Instead I shall explain the origin of this identity coming from our study of deformation of assositive submanifolds in G_2-manifolds and more general, deformation of calibrated submanifolds.

4. Calibrated and complex submanifolds

Definition (Fiorenza-L-Schwachhöfer-Vitagliano, arXiv:1804.05732) Let M be a smooth manifold and $\Psi \in \Omega^l(M, TM)$. A submanifold $L^r \subset M$, where $r \geq l$, will be called a Ψ-submanifold, if $\Psi_{|L} \in \Omega^l(L^r, TL^r)$.

- Any almost complex submanifold in an almost complex manifold is a Ψ-submanifold.

- Any φ-calibrated submanifold is $\hat{\varphi}$-submanifold.

• The Lie bracket $[,]$ on $\mathfrak{g}$ is an element in $\Lambda^2(\mathfrak{g}^*) \otimes \mathfrak{g}$. Hence any Lie group is a Ψ-submanifold.

Theorem(FLSV2018) Let $\Psi \in \Omega^*(M, TM)$ be an odd degree element which is square-zero, i.e., such that $[\Psi, \Psi]^{FN} = 0$, and let L be a Ψ-submanifold. Then the cochain complex $\Omega^*(L, NL)[-1]$ carries a canonical $\mathbf{Z}_2$-graded L_∞-algebra structure. If $\deg \Psi = 1$ then this $\mathbf{Z}_2$-graded L_∞-algebra is also a $\mathbf{Z}$-graded L_∞-algebra.

Theorem (FLSV2018) Let $\varphi \in \Omega^l(M)$ be a parallel calibration on a real analytic Riemannian manifold (M, g). If L is φ-calibrated submanifold, then there is a canonical $\mathbf{Z}_2$-graded strongly homotopy Lie algebra that governs formal and smooth deformations of L in the class of φ-calibrated submanifolds.

• McLean (1998) considered only deformations of special Lagrangian, associative, coassociative and Cayley submanifolds.

- Further works on deformations of calibrated submanifolds are devoted to the smoothness and the Zariski tangent space to the moduli space of closed calibrated submanifolds that are special Lagrangian, associative, coassociative and Cayley in (tamed) almost/nearly Calabi-Yau, G_2 and Spin(7)- manifolds

- The classical deformation theory of complex submanifolds can be formulated in a similar way.

5. CR-twistor spaces over manifolds with G$_2$- and Spin(7)-structure.

- (M, g) - oriented Riemannian manifold.

- We identify $Gr^{+}(r, M)$ with decomposable unit r-vectors in $\Lambda^r TM$.

$$T_w(\Lambda^r TM) = \Lambda^r T_{\pi(w)}M \oplus T_w^{hor}(\Lambda^r TM)$$

where

$$T_w^{hor}(\Lambda^r TM) = T_{\pi(w)}M$$

Then we have

$$T_v Gr^+(r, M) = T_v Gr^+(r, T_{\pi(v)} M) \oplus T_w^{hor}(Gr^+(r, M)).$$

where

$$T_v^{hor}(Gr^+(r, M)) = T_v^{hor}(\Lambda^r TM) = T_{\pi(v)} M.$$

Let $B \subset TGr^+(r, M)$ - a distribution

$$B(v) := \{\xi \in T_v^{hor} Gr^+(r, M) |\, d\pi(\xi) \in E_v^\perp \subset T_{\pi(v)} M\}.$$

- $\chi \in \Omega^{r+1}(M, TM)$ - a VCP on (M, g).

For $v \in Gr^+(r, M)$, $w \in B(v)$ we let

$$J_B(w) := v \times w \in B(v).$$

• $(Gr^+(r, M), B, J_B)$ is a CR-twistor space over (M, g, χ).

• An (almost) CR-structure on a manifold N is a pair (B, J_B) consisting of a distribution $B \subseteq TN$ and of an almost complex structure J_B on B. An almost CR-structure (B, J_B) is said to be integrable if $[B^{1,0}, B^{1,0}] \subseteq B^{1,0}$. If (B, J_B) is integrable, then (N, B, J_B) is called a CR-manifold.

• $(Gr^+(r-1, M), B, J_{g,\chi})$ is called the CR-twistor space over (M, g, χ).

• (B, J) defines an integrable CR-structure iff the following two conditions hold two conditon

1. $[JX, JY] - [X, Y] \in \Gamma(B) \, \forall \, X, Y \in \Gamma(B)$;

2. $N_J(X, Y) = 0 \, \forall \, X, Y \in \Gamma(B) \iff$

$$[JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0.$$

• LeBrun (1984) and Rossi (1985): the CR-twistor space over (M^n, g) with $(n-1)$-VCP is CR-manifold.

• Lempert, LeBrun (1993): the CR-integrability implies the (weak) formal integrability of J on the loop space over (M^n, g) with $(n-2)$-fold VCP, (if (M, g) is analytic).

• Verbitsky (2011): the CR-twistor space over a Riemannian manifold (M^7, φ) is integrable iff $\nabla\varphi = 0$. (This is used by Verbitsky later for his proof of the formal integrability of J on loop space over G_2-manifolds.)

Theorem (Fiorenza-L., 2021) (1) The 1st integrability for the CR-twistor space over G_2 and Spin(7)-manifolds (M, g) holds, iff (M, g) is of constant curvature.

(2) The CR-twistor space over $S^7 = \mathrm{Spin}(7)/G_2$ endowed with an Spin(7)-invariant associative 3-form is a CR-manifold.

Thank you for your attention!