On the multiple fractional integrals defined on the product of nonhomogeneous measure spaces

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- Let (X, d, μ) be a space nonhomogeneous type, i. e. a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \times X \longrightarrow R_+$ satisfying the conditions:
  (i) $d(x, x) = 0$ for all $x \in X$;
  (ii) $d(x, y) > 0$ for all $x \neq y$, $x, y \in X$;
  (iii) there exists a positive constant a_0 such that $d(x, y) \leq a_0 d(y, x)$ for every $x, y \in X$;
  (iv) there exists a positive constant a_1 such that

$$d(x, y) \leq a_1 \left(d(x, z) + d(z, y) \right) \quad \text{for every} \quad x, y, z \in X;$$

  (v) for every neighbourhood N of the point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : \ d(x, y) < r\}$ is contained in N;
  (vi) the ball $B(x, r)$ is measurable for every $x \in X$ and for arbitrary $r > 0$.

- Let

$$I^\gamma f(x) = \int_X (d(x,y))^{\gamma-1} f(x,y)\, d\mu, \quad 0 < \gamma < 1.$$

In V. Kokilashvili and A. Meskhi, Fractional integrals on measure spaces. *Fract. Calc. Appl. Anal.*, **4**(2001), No 1, 1-24 (see also D. E. Edmunds, V. Kokilashvili and A. Meskhi, Bounded and compact integral operators. Vol. 513 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrucht, 2002, Chapter 6) the following statement is proved:

Theorem A. *Let* $1 < p < q < \infty$ *and let* $0 < \gamma < 1$*. The operator* I^γ *acts boundedly from* $L^p_\mu(X)$ *to* $L^q_\mu(X)$ *if and only if there exists a constant* $c > 0$ *such that*

$$\mu B(x,r) \le c r^\beta, \quad \beta = \frac{pq(1-\gamma)}{pq+p-q}$$

for arbitrary ball $B(x,r)$*.*

- Let now $(X_j, d_j, \mu_j$ $(j = 1, 2, \dots, n)$ be the measure quasi-metric spaces. Assume that $\overrightarrow{p} = (p_1, \dots, p_n)$, $1 < p_j < \infty$ $(j = 1, 2, \dots, n)$ and $\overrightarrow{\mu} = (\mu_1, \mu_2, \mu \dots, \mu_n)$. For measurable $f : \prod_{j=1}^{n} X_j \longrightarrow R^1$ we set the mixed norm Lebesgue spaces $L^{\overrightarrow{p}}_{\overrightarrow{\mu}}\left(\prod_{j=1}^{n} X_j, \prod_{j=1}^{n} \mu_j\right)$ with the norm

$$\|f\|_{L^{\overrightarrow{p}}_{\overrightarrow{\mu}}} =$$

$$\left(\int_{X_1} \cdots \left(\int_{X_{n-1}} \left(\int_{X_n} |f(x_1, \dots, x_n)|^{p_n} d\mu_n\right)^{\frac{p_{n-1}}{p_n}} d\mu_{n-1}\right)^{\frac{p_{n-2}}{p_{n-1}}} \cdots d\mu_1\right)^{\frac{1}{p_1}}.$$

The mixed-norm Lebesgue spaces were introduced and studied in A. Benedek and R. Panzone, The spaces L^P, with mixed norm. *Duke Math. J.* **28**(1961), No 3, 301-324.

- Consider the multiple fractional integral defined on the product space $X = X_1 \times \cdots \times X_n$:

$$I^{\gamma} f(x) = \int_X \frac{f(y_1, \ldots, y_n) d\mu_1 \cdots d\mu_n}{\prod_{j=1}^{n} \left(d_j(x_j, y_j)\right)^{1-\gamma j}}, \quad \gamma = (\gamma_1, \ldots, \gamma_n). \qquad (1)$$

The following statement is true.

Theorem

Let $1 < p_j < q_j < \infty$ $(j = 1, 2, \ldots, n)$. The operator I^{γ} is bounded from $L^{\vec{p}}_{\vec{\mu}}$ to $L^{\vec{q}}_{\vec{\mu}}$ if and only if there exists a positive constant c such that

$$\mu_j B_j(x_j, r_j) \le c r_j^{\frac{p_j q_j (1-\gamma_j)}{p_j q_j + p_j - q_j}}, \quad j = 1, 2, \ldots, n \qquad (2)$$

for arbitrary balls B_j from X_j.

- Theorem says that if the condition (2) fails then I^γ is unbounded from $L_{\vec{\mu}}^{\vec{p}}$ to $L_{\vec{\mu}}^{\vec{q}}$. Nevertheless there exists a weight $\vec{v} : X \longrightarrow R^1$ such that I^γ is bounded from $L_{\vec{\mu}}^{\vec{p}}$ to $L_{\vec{\mu}}^{\vec{q}}(\vec{v})$.
  Let us introduce the functions:

$$\Omega(x_j) = \sup_{r_j>0} \frac{\mu_j B(x_j, r_j)}{r_j^{\beta_j}},$$

  where

$$\beta_j = \frac{p_j q_j (1-\gamma_j)}{p_j q_j + p_j - q_j}. \qquad (3)$$

- The following statement holds.

Theorem

Let $1 < p_j < q_j < \infty$ $(j = 1, 2, \ldots, n)$. *Then there exists a positive constant* $c > 0$ *such that for arbitrary* $f \in L_{\mu}^{\vec{p}}(X)$ *we have*

$$\left\| I^{\gamma} f(x_1, \ldots, x_n) \prod_{j=1}^{n} \Omega_j^{\frac{\gamma_j - 1}{p_j}}(x_j) \right\|_{L_{\vec{\mu}}^{\vec{q}}} \leq c \|f\|_{L_{\vec{\mu}}^{\vec{p}}}.$$

- Let now $\Gamma_i = \{t \in \mathbb{C} : \ t = t(s), 0 \leq s \leq l\}$ be arbitrary rectifiable simple curves with arc-length measures ν_i $(i = 1, 2, \dots, n)$. Suppose
$$D_j(t_j, r_j) = \Gamma_j \bigcap B_i(t_j, r_j)$$
where
$$B_i(t_i, r_i) = \{z_i \in \mathbb{C} : \ |z_j - t_j| < r_j\}, \quad t_j \in \Gamma_j.$$
Let
$$\Omega_j(t_j) = \sup_{r_j > 0} \frac{\nu_j D(t_j, r_j)}{r_j^{\beta_j}}$$
where β_j are defined by (3).

- Then for the operator

$$I_{\Gamma}^{\gamma} f(t_1, t_2, \dots, t_n) = \int\limits_{\Gamma} \frac{f(\tau_1, \tau_2, \dots, \tau_n) d\nu_1 \dots d\nu_n}{\prod\limits_{j=1}^{n} |t_j - \tau_j|^{1-\gamma_j}}, \quad \Gamma = \Gamma_1 \times \dots \times \Gamma_n$$

we have the following assertion

Theorem

Let $1 < p_j < q_j < \infty$. Then there exists a positive constant c such that for arbitrary $f \in L_{\vec{\nu}}^{\vec{p}}(\Gamma)$ we have

$$\| I_{\Gamma}^{\gamma} f(t_1, t_2, \dots, t_n) \cdot \Omega_j^{\frac{\gamma_j - 1}{p_j}}(t_j) \|_{L_{\vec{\nu}}^{\vec{q}}(\Gamma)} \le c \| f \|_{L_{\vec{\nu}}^{\vec{p}}}.$$

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