On a weighted inequality for fractional integrals

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Portorož, June 23, 2021

Abstract

We establish necessary and sufficient condition on a non-negative locally integrable function v guaranteeing the (trace) inequality

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \le C\|f\|_{L^{p,1}(\mathbb{R}^n)}$$

for the Riesz potential I_α, where $L^{p,1}(\mathbb{R}^n)$ is the Lorentz space. The same problem is studied for potentials defined on spaces of homogeneous type.

Trace inequalities for Riesz potentials I_α deals with non-negative measures ν such that

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}. \qquad (0.1)$$

D. Adams [1] proved that necessary and sufficient condition on ν guaranteeing (0.1) for $1 < p < q < \infty$ and $0 < \alpha < n/p$ is that measure ν satisfies the condition: there is a positive constant C such that for all balls $B \subset \mathbb{R}^n$,

$$\nu(B) \le C|B|^{(\frac{\alpha}{n} - \frac{1}{p})q}.$$

Introduction

Riesz potential operator

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n,$$

plays an important role in PDEs. It is worth mentioning its role in the theory of Sobolev's embeddings (see, e.g., V. G. Maz'ya [12]).

The appropriate fractional maximal operator is given by the formula:

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy, \;\; 0 \le \alpha < n, \;\; x \in \mathbb{R}^n.$$

$M_0 f = Mf$ is the Hardy–Littlewood maximal function having great importance in Harmonic Analysis for example, in the theory of Singular integrals.

Introduction

Let v be a non-negative locally integrable function on $\mathbb{R}^n$. We are interested in the inequality (0.1) for $d\nu = vdx$, i.e.

$$\|I_\alpha f\|_{L^q_v(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{0.2}$$

Introduction

In this case by the result of D. Adams [1] the condition

$$[v]_{p,q,\alpha} := \sup_B \Big(v(B)\Big)^{1/q} |B|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty, \tag{0.3}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, is simultaneously necessary and sufficient whenever $1 < p < q < \infty$ and $0 < \alpha < n/p$. In the case $p = q$ the implication (0.2) $\Rightarrow$ (0.3) can be checked easily by considering the test functions χ_B; however the fact that (0.3) $\Rightarrow$ (0.2) is not true (see appropriate counterexamples in D. R. Adams [2], R. Kerman and E. Sawyer [14] for a measure ν, and P. G. Lemarié-Rieusset [8] for non-negative function v).

Introduction

Our aim is to find a Lorentz space $L^{p,s}$, which is narrower than the class $L^p(\mathbb{R}^n)$ (i.e., $s < p$) and for which the inequality

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C\|f\|_{L^{p,s}(\mathbb{R}^n)} \tag{0.4}$$

holds if and only if (0.3) is satisfied for $p = q$. In particular we show that (0.4) is equivalent to the condition (0.3) for $s = 1$. The question for $1 < s < p$ remains open.

It should be mentioned that there are known various different criteria for (0.2) with $p = q$ (see D. R. Adams [2], V. G. Maz'ya [10], V. G. Maz'ya [11], R. Kerman and E. Sawyer [14], V. G. Maz'ya and I. Verbitsky [13]). For the solution of the two-weight problem for Riesz potential operators I_α we refer to M. Gabidzashvili and V. Kokilashvili [6], E. Sawyer [15] (see also the monograph V. Kokilashvili and M. Krbec [7]).
Inequality (0.2) for $p = q$ implies the estimate:

$$\|f\|_{L^q_v(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty, \tag{0.5}$$

which follows from the estimate

$$|f(x)| \leq CI_1(|\nabla f|)(x).$$

Introduction

The following Fefferman-Phong C. Fefferman [4] type theorem holds:

Theorem (A)

Let $1 < p < \infty$ and let $0 < \alpha < n/p$. Then the following inequality holds:

$$\|I_\alpha f\|_{L^p_v} \leq C[v]^*_{p,r,\alpha}\|f\|_{L^p}$$

for some $p < r$, where

$$[v]^*_{p,r,\alpha} := \sup_B |B|^{\frac{\alpha}{n}-\frac{1}{r}}\left(\int_B v^{r/p}(x)dx\right)^{1/r} < \infty. \qquad (0.6)$$

Remark 1: It is easy to see that by Hölder's inequality we have that condition (0.6) is stronger than (0.3) for $p = q$, in particular, $[v]_{p,\alpha} \leq [v]^*_{p,r,\alpha}$ for $r > p$, where $[v]_{p,\alpha} = [v]_{p,p,\alpha}$.

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \le p < \infty$, $1 \le s \le \infty$. We say that f belongs to the Lorentz space $L^{p,s}$ if

$$\|f\|_{L^{p,s}} = \begin{cases} \left(s \int\limits_0^\infty \left(|\{x \in \mathbb{R}^n : |f(x)| > \tau\}| \right)^{s/p} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \le s < \infty, \\ \sup\limits_{s>0} s \Big(|\{x \in \mathbb{R}^n : |f(x)| > s\}| \Big)^{1/p}, & \text{if } s = \infty \end{cases}$$

is finite.

If $p = s$, then $L^{p,s}$ coincides with the weighted Lebesgue space L^p.

It is worth mentioning, that if $1 \le p < \infty$, $s_2 \le s_1$, then $L^{p,s_2} \hookrightarrow L^{p,s_1}$ with the embedding constant C_{p,s_1,s_2} depending only on p, s_1 and s_2;

Main Result

Theorem (1)

Let $1 < p < \infty$ and let $0 < \alpha < n/p$. Then the following statements are equivalent:

(i) there is a positive constant C such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C\|f\|_{L^{p,1}(\mathbb{R}^n)} \tag{0.7}$$

(ii) there is a positive constant c such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|M_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq c\|f\|_{L^{p,1}(\mathbb{R}^n)} \tag{0.8}$$

(iii) $[v]_{p,\alpha} = \sup_B \big(v(B)\big)^{1/p}|B|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty.$

Moreover, if C and c are best constant in (0.7) and (0.8) respectively, then

$$C \approx c \approx [v]_{p,\alpha}.$$

The case of Spaces of Homogeneous Type

Let (X, d, μ) be a quasi-metric measure space with a quasi-metric d and measure μ. A quasi-metric d is a function $d\colon X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) for all $x, y \in X$, $d(x, y) = d(y, x)$;

(iii) there is a positive constant κ such that

$$d(x, y) \le \kappa \left(d(x, z) + d(z, y) \right)$$

for all $x, y, z \in X$.

The case of Spaces of Homogeneous Type

In what follows we will assume that the balls $B(x, r) := \{y \in X;\ d(x, y) < r\}$ are measurable with positive μ measure for all $x \in X$ and $r > 0$.
If μ satisfies the doubling condition:

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)), \tag{0.9}$$

with a positive constant C_μ independent of x and r, then we say that (X, d, μ) is a space of homogeneous type (SHT). We will assume that (X, d, μ) is an SHT.

The case of Spaces of Homogeneous Type

For example, rectifiable curves in $\mathbb{C}$ with Euclidean distance and arc-length measure satisfying Carleson (regularity) condition, nilpotent Lie groups with Haar measure, domains in $\mathbb{R}^n$ with so-called $\mathcal{A}$ condition are examples of an SHT. For the definition, examples and some properties of an SHT see, e.g., the paper R. A. Macías and C. Segovia [9] and the monographs J. O. Strömberg and A. Torchinsky [16], R. R. Coifman and G. Weiss [3].

For a given quasi-metric measure space (X, d, μ) and q satisfying $1 \leq q \leq \infty$, as usual, we will denote by $L^q = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. Let $L^{p,s}(X, \mu)$ be the Lorentz space defined on an SHT (X, d, μ).

Let us denote by $K_\alpha f$ Riesz potential of a $\mu-$ measurable function f given by the formula:

$$K_\alpha f(x) = \int_X \mu(B_{xy})^{\alpha-1} f(y)\, d\mu(y), \quad x \in X,$$

where $0 < \alpha < 1$, $B_{xy} := B(x, d(x,y))$.
The appropriate fractional maximal function has the form

$$\mathcal{M}_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad x \in X.$$

The case of Spaces of Homogeneous Type

The following trace inequality for an *SHT* was proved by Gabidzashvili (see [5]).

Theorem (B)

Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an *SHT* and ν is another measure on X.Then the inequality

$$\|K_\alpha f\|_{L^q(X,\nu)} \leq C\|f\|_{L^p(X,\mu)}$$

holds if and only if

$$\sup_B \Big(\nu B\Big)^{1/q} \mu(B)^{\alpha-\frac{1}{p}} < \infty.$$

Analyzing the proof of Theorem (1) we can formulate the same result for an *SHT*. In particular, the following Theorem holds:

The case of Spaces of Homogeneous Type

Theorem (2)

Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) be an SHT. Assume that v is non-negative μ locally integrable function on X. Then the following statements are equivalent:

(i) there is a positive constant C such that for all $f \in L^{p,1}(X, \mu)$,

$$\|K_\alpha f\|_{L^p_v(X,\mu)} \leq C\|f\|_{L^{p,1}(X,\mu)}; \tag{0.10}$$

(ii) there is a positive constant c such that for all $f \in L^{p,1}(X, \mu)$,

$$\|\mathcal{M}_\alpha f\|_{L^p_v(X,\mu)} \leq c\|f\|_{L^{p,1}(X,\mu)}; \tag{0.11}$$

(iii) $[v]_{p,\alpha,X,\mu} = \sup\limits_B \left(\int\limits_B v(x) d\mu(x) \right)^{1/p} \mu(B)^{\alpha - \frac{1}{p}} < \infty.$

Moreover, if C and c are best constants in (0.10) and (0.11) respectively, then $C \approx c \approx [v]_{p,\alpha,X,\mu}$.

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Acknowledgement

The work was supported by the Shota Rustaveli National Science Foundation of Georgia (Project No. DI-18-118).

Thank you for your attention!