Gauss-Lucas theorem in polynomial dynamics

Margaret Stawiska-Friedland

AMS/Mathematical Reviews, Ann Arbor, MI, USA

8th ECM Portorož June 2021
Minisymposium on Approximation Theory and Applications

Complex dynamics

Consider a rational function f of degree $d > 1$, i.e.,

$$f(z) = \frac{P(z)}{Q(z)},\ P, Q \in \mathbb{C}[z],\ d := \max\{\deg P, \deg Q\} > 1.$$

Complex dynamics

Consider a rational function f of degree $d > 1$, i.e.,

$$f(z) = \frac{P(z)}{Q(z)},\ P, Q \in \mathbb{C}[z],\ d := \max\{\deg P, \deg Q\} > 1.$$

Such a function extends naturally as a holomorphic map to $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.
Complex dynamics studies behavior of the sequence of iterates $f^{\circ(n+1)} = f \circ f^{\circ n},\ (n \in \mathbb{N})$ in $\mathbb{C}_\infty$.

The Julia-Fatou dichotomy

We define the Fatou set $\mathcal{F}_f$ of f and the Julia set $\mathcal{J}_f$ of f as follows: $\mathcal{F}_f$ is the maximal open subset of $\mathbb{C}_\infty$ on which the sequence $\{f^{\circ n} : n \in \mathbb{N}\}$ is equicontinuous, and $\mathcal{J}_f$ is the complement of $\mathcal{F}_f$ in $\mathbb{C}_\infty$.

The Julia-Fatou dichotomy

We define the Fatou set $\mathcal{F}_f$ of f and the Julia set $\mathcal{J}_f$ of f as follows: $\mathcal{F}_f$ is the maximal open subset of $\mathbb{C}_\infty$ on which the sequence $\{f^{\circ n} : n \in \mathbb{N}\}$ is equicontinuous, and $\mathcal{J}_f$ is the complement of $\mathcal{F}_f$ in $\mathbb{C}_\infty$. The Julia set is closed and nonempty, in fact infinite. Both Julia and Fatou sets are totally f-invariant, i.e.,

$$f^{-1}(J(f)) = J(f) = f(J(f)),\ f^{-1}(F(f)) = F(f) = f(F(f)).$$

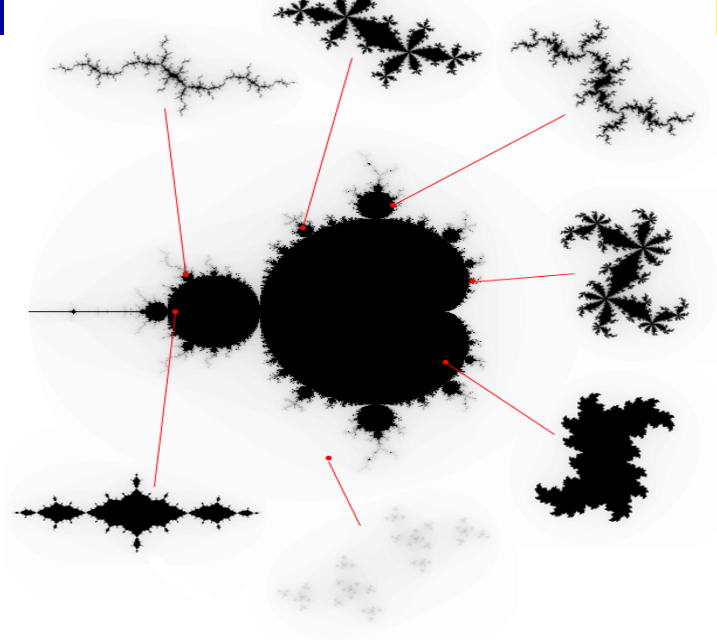
Dynamics of polynomials, I

For a polynomial p (of degree $d > 1$), the point $\infty = p(\infty) = p^{-1}(\infty)$ belongs to the Fatou set. Further, the Julia set $\mathcal{J}_p \subset \mathbb{C}$ always has empty interior.

Dynamics of polynomials, I

For a polynomial p (of degree $d > 1$), the point $\infty = p(\infty) = p^{-1}(\infty)$ belongs to the Fatou set. Further, the Julia set $\mathcal{J}_p \subset \mathbb{C}$ always has empty interior.

Some examples of Julia sets of quadratic polynomials can be seen in the next slide.



Dynamics of polynomials, II

Since

$$\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^d} > 0,$$

there exists an $R > 0$ such that $p^{-1}(D_R) \subset D_R$, where $D_R := \{z : |z| \leqslant R\}$. Furthermore, for any such R and for each positive integer k_0 we have

$$\emptyset \neq \mathcal{K}_p = \bigcap_{k \geqslant k_0} p^{-k}(D_R),$$

where $\mathcal{K}_p := \{z \in \mathbb{C} : \{p^{\circ n}(z)\}\text{is bounded}\}$. We call $\mathcal{K}_p$ the **filled-in Julia set** of p. It is easy to show that $p^{-1}(\mathcal{K}_p) = \mathcal{K}_p = p(\mathcal{K}_p)$ and that $\mathcal{K}_p$ is the union of $\mathcal{J}_p = \partial\mathcal{K}_p$ with bounded components of $\mathcal{F}_p$.

A natural question

Do they exist nontrivial closed sets $K \subset \mathbb{C}$ (other than $\mathcal{J}_p$, $\mathcal{K}_p$ or D_R) containing $\mathcal{J}_p$ such that $p^{-1}(K) \subset K$?

A natural question

Do they exist nontrivial closed sets $K \subset \mathbb{C}$ (other than $\mathcal{J}_p$, $\mathcal{K}_p$ or D_R) containing $\mathcal{J}_p$ such that $p^{-1}(K) \subset K$?
More specifically, for a complex polynomial p of degree $d \geqslant 2$, let $H_p = \mathrm{conv} J_p$ be the convex hull of the Julia set of p. Do we always have $p^{-1}(H_p) \subset H_p$?

A natural question

Do they exist nontrivial closed sets $K \subset \mathbb{C}$ (other than $\mathcal{J}_p$, $\mathcal{K}_p$ or D_R) containing $\mathcal{J}_p$ such that $p^{-1}(K) \subset K$?
More specifically, for a complex polynomial p of degree $d \geqslant 2$, let $H_p = \mathrm{conv} J_p$ be the convex hull of the Julia set of p. Do we always have $p^{-1}(H_p) \subset H_p$?
This was conjectured by P. Alexandersson and answered positively by the present author.

A relation between convex sets and complex polynomials

Theorem

(Gauss-Lucas theorem) Every convex set in the complex plane containing all the zeros of a complex polynomial p also contains all critical points of p (solutions to $p'(z) = 0$).

A relation between convex sets and complex polynomials

Theorem

(Gauss-Lucas theorem) Every convex set in the complex plane containing all the zeros of a complex polynomial p also contains all critical points of p (solutions to $p'(z) = 0$).

The following result due to W. P. Thurston is equivalent to the Gauss-Lucas theorem:

Theorem

Let p be any polynomial of degree at least two. Denote by $\mathcal{C}$ the convex hull of the critical points of p. Then $p : E \to \mathbb{C}$ is surjective for any closed half-plane E intersecting $\mathcal{C}$.

Other useful results

Theorem

(hyperplane separation theorem) Let X be a convex and closed subset of a finite-dimensional vector space V. If $x_0 \notin X$, then there is an affine half-space containing x_0 which does not intersect X; that is, there is an affine function $f : V \to \mathbb{R}$ with $f(x_0) < 0 \leqslant f(x),\ x \in X$.

Other useful results

Theorem

(hyperplane separation theorem) Let X be a convex and closed subset of a finite-dimensional vector space V. If $x_0 \notin X$, then there is an affine half-space containing x_0 which does not intersect X; that is, there is an affine function $f : V \to \mathbb{R}$ with $f(x_0) < 0 \leqslant f(x),\ x \in X$.

Lemma

(a consequence of Gauss-Lucas due to L. Hörmander) Let p be a complex polynomial and let B be a closed convex subset of $\mathbb{C}$ containing all zeros of p'. Then the set C_B of all $w \in \mathbb{C}$ such that all the zeros of $p(\cdot) - w$ are contained in B is a convex set.

“Dynamical Gauss-Lucas”

We will prove Alexandersson’s conjecture using the following :

Lemma

Let p be any polynomial of degree at least two. Then all zeros of p' belong to $H_p = \mathrm{conv} J_p$.

Proof.

Suppose there is an $x_0 \notin H_p$ such that $p'(x_0) = 0$. By the hyperplane separation theorem (applied twice if necessary), there exists a closed half-plane E such that $x_0 \in E$ and $E \cap J_p = \emptyset$. By Thurston’s theorem, $p : E \to \mathbb{C}$ is surjective. Take a $z_0 \in J_p$. Then on one hand $p^{-1}(z_0) \subset J_p$, while on the other hand $p^{-1}(z_0) \cap E \neq \emptyset$, a contradiction. □

The main result

Theorem

Let p be a complex polynomial of degree $d \geqslant 2$. Then $p^{-1}(H_p) \subset H_p$.

Proof.

By "dynamical Gauss-Lucas", $B = H_p$ satisfies the assumptions of Hörmander's Lemma. Hence the set $C_p = \{w \in \mathbb{C} : p^{-1}(w) \in H_p\}$ is convex. Furthermore, for $w \in J_p$ we have $p^{-1}(w) \in J_p \subset H_p$, so $J_p \subset C_p$. Hence $H_p \subset C_p$, which implies $p^{-1}(H_p) \subset H_p$. □

Further results

We can further prove that the equality $p^{-1}(H_p) = H_p$ is achieved if and only if J_p is either a line segment or a circle; that is, if and only if p is Möbius conjugated to the classical Chebyshev polynomial T_d of degree d, to $-T_d$ or the monomial cz^d with $|c| = 1$.

Further results

We can further prove that the equality $p^{-1}(H_p) = H_p$ is achieved if and only if J_p is either a line segment or a circle; that is, if and only if p is Möbius conjugated to the classical Chebyshev polynomial T_d of degree d, to $-T_d$ or the monomial cz^d with $|c| = 1$.
The distinction is according to whether $J_p = K_p$ or $J_p \subsetneq K_p$.
Note that K_p is the holomorphically convex hull of J_p in $\Omega = \mathbb{C}$.

The case of a segment

Let p be a complex polynomial of degree $d \geqslant 2$ such that $H_p = p^{-1}(H_p) = J_p$. Then J_p is a line segment.

Proof.

Recall that for any polynomial p the Julia set J_p has empty interior. If $J_p = H_p$, then J_p is a closed convex set in $\mathbb{C}$ with empty interior, and hence it is a subset of a line. Being connected and compact, it must be a (closed) segment. □

THANK YOU FOR YOUR ATTENTION!

Bibliography

- P. Alexandersson, *Convex Julia sets*, Math Overflow question, April 2, 2020, `https://mathoverflow.net/questions/356342/convex-julia-sets`
- A. Chéritat, Y. Gao, Y. Ou, L. Tan, *A refinement of the Gauss-Lucas theorem (after W. P. Thurston)*, C. R. Math. Acad. Sci. Paris 353 (2015), no. 8, 711-715.
- L. Hörmander, *Notions of Convexity*, Springer Science & Business Media, 2007 (Modern Birkhäuser Classics) ISBN 0817645853, 9780817645854
- M. Stawiska, *Convex hulls of polynomial Julia sets*, Proceedings of the AMS, Vol. 149, No. 1, January 2021, pp. 245-250 https://doi.org/10.1090/proc/15224