Subclasses of Partial Contraction Mapping and generating evolution system on Semigroup of Linear Operators

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Abstract

Some properties of C_0-Semigroup are investigated and used to derive properties of ω-Order Preserving and Reversing Partial Contraction Mapping where homogeneous, inhomogeneous and regularity of mild solution for analytic semigroups are engaged. Furthermore, the subclasses performed like semigroup of linear operators. Moreover, semigroup of linear operator generated by ω-order reversing partial contraction mapping (ω-$ORCP_n$) as the infinitesimal generator of a C_0-semigroup is discussed. It is an attempt to obtain results on evolution systems and stable families of generators considering the homogeneous and inhomogeneous initial value problem.

1 Introduction

Recently, ω-Order semigroup was introduced and was established as subset of C_0-semigroup. Let X be a Banach space and K be a linear operator such that $K : D(K) \subseteq X \to X$. Given $x \in X$, the abstract Cauchy problem for operator K with initial data x comprised of finding a solution $u(t)$ to the homogeneous, inhomogeneous and regularity of mild solution for analytic semigroups.

The homogeneous Initial Value Problem (IVP)

$$\left\{\begin{array}{ll} \frac{du(t)}{dt} = Au(t) & t > 0 \\ u(0) = x & \end{array}\right\} \tag{1.1}$$

where the solution means an X valued function $u(t)$ such that $u(t)$ is continuous for $t \geq 0$, continuously differentiable and $u(t) \in D(K)$ for $t > 0$ such that (1.1) is valid. Obviously, $u(t) \in D(K)$ for $t > 0$ and u is continuous at $t = 0$, (1.1) can not have a solution for $x \notin \overline{D(K)}$. It was proved that if operator K in ω-Order preserving (OCP_n) or ω-Order reversing ($ORCP_n$) partial Contraction mapping then K is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ which is a semigroup of linear operator. The abstract Cauchy problem for K has a solution $u(t) = T(t)x$, for every $x \in D(K)$.

In the case of inhomogeneous IVP,

$$\left\{\begin{array}{ll} \frac{du(t)}{dt} = Au(t) + f(t) & t > 0 \\ u(0) = x & \end{array}\right. \tag{1.2}$$

where $f : [0, T] \rightarrow X$. In this article, K is assumed to be infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ such that the corresponding homogeneous equation (equation with $f \equiv 0$) contains a unique solution for every initial value $x \in D(K)$ and $f \in L^1([0, T]; X)$. Furthermore, we shall take the regularity of the mild solutions for analytic semigroups except otherwise stated. The mild solution of the IVP (1.2) is the continuous function

$$u(t) = T(t)x + \int_0^t T(t-s)(s)ds. \tag{1.3}$$

By imposing further conditions on f ($f \in C^1([0, T]; X)$), the mild solution (1.3) becomes the classical solution, hence, a continuously differentiable solution of (1.2). Suppose K is the infinitesimal generator of an analytic semigroup, then the results imply that $T(t)$ is an analytic semigroup with $f \in L^p([0,T];X)$ and $p > 1$, hence (1.3) is Hölder continuous. The theory of stability is very important since stable C_0-semigroup correspond one-to-one to asymptotically stable (in the sense of Lyapunov) in a well-posed abstract linear Cauchy problems. The resolvent of K can be used to describe the relationship between the spectrum of K and that of semigroup operator $(T(t)_{t\geq 0})$ and to establish the relationship between a semigroup operator, its generator and its resolvent.

A significant aspect of C_0-semigroup is dual properties of a semigroup of linear operator because of the emphasis on weakly topologies of operator that makes it to obtain a weak generator of a semigroup $(T(t)^*)_{t\geq 0}$.

Several authors established results on the theory of semigroups of operator, see [1]-[11] and the reference therein. This paper consists of results on homogeneous, inhomogeneous and the regularity of mild solution for analytic semigroups of bounded linear operator. Let X be a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - ORCP_n$ the ω-order reversing partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ be the spectrum and $A \in \omega - ORCP_n$ is a generator of C_0-semigroup. This paper consist of results of evolution system considering the homogeneous and inhomogeneous initial value problem of bounded linear operator. Balakrishnan [1], obtained an operator calculus for infinitesimal generators of semigroup. Agmon *et al.* [2], estimated some boundary problems for solutions of elliptic partial differential equation. Banach [3], established and introduced the concept of Banach spaces. Brezis and Gallouet [4], investigated nonlinear Schrodinger evolution equation. Chill and Tomilov [5], deduced some analytic functions and stability of operator semigroups and in [6] established some resolvent approach to stability operator semigroup. Engel and Nagel [7], obtained one-parameter semigroup for linear evolution equations. Pazy [8], introduced asymptotic behavior of the solution of an abstract evolution and some applications and also in [9], established a class of semi-linear equations of evolution. Prüss [10], proves some semilinear evolution equations in Banach spaces. Rauf and Akinyele [11], obtained ω-order-preserving partial contraction mapping and established its properties, also in [12], Rauf *et al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [13], deduced some results of C_0-semigroup and its applications. Walker [14], presented some dynamical systems and evolution. Yosida [15], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2 Preliminaries

We recall the following definitions, provide some examples and present an elementary prove of a known result.

Definition 2.1 (C_0-Semigroup) [13]: A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [11]: A transformation $\alpha \in P_n$ is called ω-order-preserving Partial Contraction Mapping if $\forall x, y \in$Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that

$T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (ω-$ORCP_n$) [11]: A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.4 (Analytic Semigroup) [13]: We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:

(i) $T(t) = S(t)$ for each $t \geq 0$;

(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;

(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and

(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, and if the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 2.5 (Classical Solution) [7]: A function $u : [0, T] \to X$ is a classical solution of (1.2) on $[0, T]$ if u is continuous and continuously differentiable on $[0, T]$, $u(t) \in D(A)$ for $0 < t < T$ and (1.2) is satisfied on $[0, T]$.

Definition 2.6 (Compact Semigroup) [7]: A C_0-semigroup is compact if for each $t > 0$, $T(t)$ is a compact operator.

Definition 2.7 (Dissipative) [13]: A linear operator $(A, D(A))$ is dissipative if each $x \in X$ there exists $x^* \in F(x)$ such that $Re(Ax, x^*) \leq 0$.

Definition 2.8 (Hölder Continuity) [13]: A real or complex-valued function f on d-dimensional Euclidean space satisfies a Hölder condition, or is Hölder continuous, when there are non-negative real constants C, $\alpha > 0$, such that

$$|f(x) - f(y)| \leq C\|x - y\|^{\alpha}$$

for all x and y in the domain of f. More generally, the condition can be formulated for functions between any two metric spaces. The number α is called the exponent of the Hölder condition. A function on an interval satisfying the condition with $\alpha > 1$ is constant. If $\alpha = 1$, then the function satisfies a Lipschitz condition. For any $\alpha > 0$, the condition implies that the function is uniformly continuous.

Definition 2.9 (Locally Hölder Continuous) [7]: Let I be an interval. A function $f : I \to X$ is Hölder continuous with exponent $\zeta : 0 < \zeta < 1$ on I if there is a constant L such that

$$\|f(t) - f(s)\| \leq L|t - s|^{\zeta} \quad for\ s, t \in I. \tag{2.1}$$

It is locally Hölder continuous if every $t \in I$ has a neighborhood in which f is Hölder continuous. It is easy to check that if I is compact, then f is Hölder continuous and locally Hölder continuous on I. We denote the family of all Hölder function with exponent ζ on I by $C^{\zeta}(I : X)$.

Definition 2.4 (Evolution System) [7]

A two parameter family of bounded of a bounded linear operators $U(t, s)$, $0 \leq s \leq t \leq T$ on X is called an *evolution system* if the following conditions are satisfied:

(i) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$; and

(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Definition 2.5 (Stable) [7]

Let X be a Banach space. A family $(A(t)^*)_{t\in[0,T]}$ of infinitesimal generators of C_0-semigroup on X is called *stable* if there exists a constant $M \geq 1$ and ω (called the stability constants) such that

$$\rho(A(t)) \supset (\omega, \infty) \quad for\ t \in [0,T] \tag{2.2}$$

and

$$\Big\| \prod_{i=1}^{k} R(\lambda; A_1) \Big\| \leq M(\lambda - \omega)^{-k} \quad for \lambda > \omega \tag{2.3}$$

and every finite sequence $0 \leq t_1 \leq t_2, ..., t_k \leq T$, $k = 1, 2, ...$

Example 1: For any 3×3 matrix $[M_m(\mathbb{C})]$ and for each $\lambda > 0$ such that $\lambda \in \rho(K)$ where $\rho(K)$ is a resolvent set on X, if

$$K = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} then \quad T(t) = \begin{pmatrix} e^{t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \end{pmatrix} = e^{tK\lambda}.$$

Example 2: By the translation semigroup starting from $Kf = f'$ on $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, the operator $K^2 f = f''$ generates a bounded analytic semigroup. A slightly more involved case of several space dimensions is the spaces $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. Denoted by $(\cup_i(t))_{t\in\mathbb{R}_+}$ is the strongly continuous semigroup $(\cup_i(t)f)(x) = f(x_1, \cdots, x_{i-1}, x_i + t, \cdots, x_n)$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $1 \leq i \leq n$, where Ki is its generator and $K \in \omega - OCP_n$. Obviously, these semigroups commutes as the resolvent of Ki and hence of $K^2 i$.

Example 3: Suppose $K : D(K) \subseteq X \to X$ is an unbounded generator of a strongly continuous semigroup and take an isomorphism $S \in L(X)$ such that

$D(K) \cap S(D(K)) = \{0\}$. Then $B = SKS^{-1}$ is a generator as well, but $K+B$ is defined only on $D(K+B) = D(K) \cap D(B) = D(K) \cap S(D(K)) = \{0\}$. A tangible example for this circumstances is on $X = C_0(\mathbb{R}_+)$ by $Kf = f'$ with its canonical domain $D(K) = C_0'(\mathbb{R}_+)$ and $Sf = q.f$ for some continuous, positive function q such that q and q^{-1} are bounded and nowhere differentiable. Defining the operator B as $Bf = q.(q^1.f)'$ on $D(B) = \{f \in X : q^{-1}.f \in D(K)\}$, we obtain that the sum $K+B$ is defined only on $\{0\}$.

2.1 Theorem [15]

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.4}$$

2.2 Theorem [7]

Let $A : D(A) \subseteq X \to X$ be a densely defined operator. Then A generates a C_0-semigroup of contractions on X if and only if

i. A is dissipatives; and

ii. there exists $\lambda > 0$ such that $\lambda I - A$ is surjective.

Moreover, if A generates a C_0-semigroup of contractions, then $\lambda I - A$ is surjective for any $\lambda > 0$, and we have $Re(Ax, x^*) \leq 0$ for each $x \in D(A)$ and each $x^* \in F(x)$.

2.3 Lemma [13]

Let $u(t)$ be a continuous X valued function on $[0, T]$, if

$$\left\| \int_0^T e^{ns} u(s) ds \right\| \leq M forn = 1, 2, \cdots \tag{2.5}$$

then, $u(t) = 0$ on $[0, T]$.

Proof:

Let $x^* \in X^*$ and set $\varphi =< x^*, u(t) > 0$, then, φ is clearly continuous on $[0, T]$ and

$$\left| \int_0^T e^{ns} \varphi(s) ds \right| = \left| < x^*, \int_0^T e^{ns} u(s) ds > \right| \leq \|x^*\|.M = M_1 \; for \, n = 1, 2, \cdots \tag{2.6}$$

We show that (2.5) implies that $\varphi(t) \equiv 0$ on $[0, T]$ and since $x^* \in X^*$ was arbitrary, it follows that $u(t) \equiv 0$ on $[0, T]$.

Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kn\tau} = 1 - exp\{-e^{n\tau}\}$. This series converges uniformly to τ on bounded intervals. Therefore,

$$\begin{aligned} &\left| \int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kn(1-T+s)} \varphi(s) ds \right| \\ \leq \sum_{k=1}^{\infty} \frac{1}{k!} e^{kn(1-T)} &\left| \int_0^T e^{kns} \varphi(s) ds \right| \leq M_1 \; (exp\{e^{n(1-t)}\} - 1) \end{aligned} \tag{2.7}$$

For $t < T$, the right - hand side of (2.4) tends to zero as $n \to \infty$. On the other hand, we have

$$\int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kn(1-T+s)} \varphi(s) ds = \int_0^T (1 - exp\{-e^{n(1-T+s)}\} \varphi(s)) ds. \tag{2.8}$$

Using Lebesque's dominated convergence theorem, we noticed that the right hand side of (2.5) converges to $\int_{T-1}^{T} \varphi(s) ds$ as $n \to \infty$. Combining this together with (2.4), we found out that for every $0 \leq t \leq T$, we have $\int_{T-1}^{T} \varphi(s) ds = 0$, which implies $\varphi(t) \equiv 0$ on $[0, T]$.

3 Main Results

This section presents homogeneous, inhomogeneous and the regularity of mild solution for analytic semigroups IVP on ω-OCP_n and ω-$ORCP_n$.

Theorem 3.1

Let $K \in OCP_n$ be densely defined linear operator. If $R(\lambda; K)$ exists for all real $\lambda \geq \lambda_0$ and

$$\lim_{\lambda \to \infty} Sup \lambda^{-1} log \|R(\lambda; K)\| = 0, \tag{3.1}$$

then, system (1.1) has at most one solution for every $x \in X$.

Proof:

We noted that $u(t)$ is a solution of (1.1) if and only if $e^{2t}u(t)$ is a solution of the IVP.

$$\frac{dv}{dt} = (K + zI)v, V(0) = x.$$

Thus, $K \in OCP_n$ can be translated by a constant multiple of the identity and assume that $R(\lambda; K)$ exists for all real λ, $\lambda \geq 0$ and that (3.1) is satisfied. Let $u(t)$ be a solution of (1.1) satisfying $u(0) = 0$. We need to show that $u(t) = 0$, to this end, consider the function $t \to R(\lambda; K)u(t)$ for $\lambda > 0$. Since $u(t)$ is a solution of (1.1), then, we have

$$\frac{d}{dt}R(\lambda; K)u(t) = R(\lambda; K)Ku(t) = \lambda R(\lambda; K)u(t) - u(t),$$

which implies

$$R(\lambda; K)u(t) = -\int_0^1 e^{\lambda(t-\tau)}(\tau)d\tau. \tag{3.2}$$

From the assumption (3.1) it follows that for every $\sigma > 0$, we have

$$\lim_{\lambda \to \infty} e^{-\sigma\lambda}\|R(\lambda; K)\| = 0,$$

and therefore it follows from (3.2) that

$$\lim_{\lambda\to\infty}\int_0^{t-\sigma} e^{\lambda(t-\sigma-\tau)}u(\tau)d\tau = 0. \tag{3.3}$$

From Lemma 2.1, we deduce that $u(\tau) \equiv 0$ for $0 \le \tau \le t-\sigma$. Since t and σ are arbitrary, then $u(t) \equiv 0$ for $t \ge 0$. The proof is complete.

Theorem 3.2

Let $K \in ORCP_n$ be densely defined linear operator with nonempty resolvent set $\rho(K)$. Then, the initial value problem (1.1) has a unique solution $u(t)$ which is continuously differentiable on $[0,\infty]$ for every initial value $x \in D(K)$ if and only if K is the infinitesimal generator of a C_0-semigroup $\{T(t); t \ge 0\}$.

Proof:

If K is the infinitesimal generator of a C_0-semigroup $\{T(t); t \ge 0\}$, then it follows that $T(t)x$ is the unique solution of (1.1) with the initial value $x \in D(K)$. Moreover, $T(t)x$ is continuously differentiable for $0 \le t < \infty$. On the other hand, if (1.1) has a unique continuously differentiable solution on $[0,\infty]$ for every initial data $x \in D(K)$, then we see that $K \in ORCP_n$ is the infinitesimal generator of C_0-semigroup $\{T(t); t \ge 0\}$. We now assume that (1.1) has a unique continuously differentiable solution on $[0,\infty]$ which is denoted by $u(t,x)$. Otherwise, for $x \in D(K)$, we define the graph norm by $|x|_G = \|x\| + \|Kx\|$. Since $\rho(K) \ne \phi$, then K is closed and therefore $D(K)$ endowed with graph norm is a Banach space which we denote by $[D(K)]$. Let X_{t_0} be a Banach space of continuous functions from $[0,t_0]$ onto $[D(K)]$ with the usual supremum norm. We consider the mapping $S : [D(K)] \to X_{t_0}$ defined by $Sx = u(t;x)$ for $0 \le t \le t_0$. From the linearity of (1.1) and the uniqueness of the solutions, it is clear that S is a linear operator defined on all of $[D(K)]$. Then, the operator S is closed. Indeed, if $x_n \to x$ in $[D(K)]$

and $Sx_n \to v$ in X_{t_0}, then from the closeness of K and

$$u(t;x_n) = x_n + \int_0^t Ku(\tau;x_n)d\tau,$$

it follows that as $n \to \infty$, we have

$$v(t) = x + \int_0^t Kv(\tau)d\tau,$$

which implies that $v(t) = u(t;x)$ and S is closed. Therefore, by the closed graph theorem, S is bounded and

$$sup_{0\leq t\leq t_0}|u(t;x)|_G \leq C|x|_G. \tag{3.4}$$

We now define a mapping $T(t) : [D(K)] \to [D(K)]$ by $T(t)x = u(t;x)$.
From the uniqueness of the solutions of (1.1), it follows that $T(t)$ has the semigroup property. From (3.4) and for $0 \leq t \leq t_0$, then, $T(t)$ is uniformly bounded. This implies that $T(t)$ can be extended by $T(t)x = T(t - nt_0)T(t_0)^n x$ for $nt_0 \leq t < (n_0+1)t$ to a semigroup on $[D(K)]$ satisfying

$$|T(t)x|_G \leq Me^{\omega t}|x|_G. \tag{3.5}$$

We now need to show that

$$T(t)Ky = KT(t)y. \tag{3.6}$$

for all $y \in D(K^2)$ and $K \in \omega\text{-}ORCP_n$. By putting

$$v(t) = y + \int_0^t u(s;Ky)ds, \tag{3.7}$$

then we have

$$v^{'}(t) = u(t;Ky) = Ky + \int_0^t \frac{d}{ds}(s;Ky)ds = K(y + \int_0^t u(s;Ky)ds) = Kv(t). \tag{3.8}$$

Since $v(0) = y$ we have by uniqueness of the solution (1.1), $v(t) = u(t; y)$ and $Kx(t; y) = v^{'}(t) = u(t; Ky)$ which is the same as (3.6). Since $D(K)$ is dense in X and by our assumption, $\rho(K) = \phi$, also $D(K^2)$ is dense in X. Let $\lambda_0 \in \rho(K)$, $\lambda_0 = 0$ be fixed and let $y \in D(K^2)$. Assume $x = (\lambda_0 I - K)y$, then by (3.6), we have

$$T(t)x = (\lambda_0 I - K)T(t)y \tag{3.9}$$

therefore

$$\|T(t)x\| = \|(\lambda_0 I - K)T(t)y\| \leq C|T(t)y|_G \leq C_1 e^{\omega t}|y|_G. \tag{3.10}$$

But

$$|y|_G = \|y\| + \|Ky\| \leq C_2\|x\|, \tag{3.11}$$

which implies that

$$\|T(t)x\| \leq C_2 e^{\omega t}\|x\|. \tag{3.12}$$

Therefore $T(t)$ can be extended to all of X by Continuity. After this extension, $T(t)$ becomes a C_0-semigroup on X. To complete the proof, we have to show that K is the infinitesimal generator of $T(t)$. Let denote the infinitesimal generator of $T(t)$ by $K_1 \in \omega\text{-}ORCP_n$. Assume $x \in D(K)$ by definition of $T(t)$, we have $T(t)x = x(t; x)$ and by assumption that

$$\frac{d}{dt}T(t)x = AT(t)x, for \;\; t \geq 0, \tag{3.13}$$

which implies, in particular, that $(d/dt)T(t)x|_{t=0} = K_x$, therefore $K_1 \supset K$. Let $Re\lambda > \omega$ and $y \in D(K^2)$. It follows from (3.6) and $K_1 \supset K$ that

$$e^{-\lambda t}AT(t)y = e^{-\lambda t}T(t)Ay = e^{-\lambda t}T(t)A_t y. \tag{3.14}$$

Integrating (3.14) from 0 to ∞ yields

$$KR(\lambda; K_1)y = R(\lambda; K_1)K_1 y. \tag{3.15}$$

But $K_1 R(\lambda; K_1)y = R(\lambda; K_1)K_1 y$. Hence, $KR(\lambda; K_1)y = R(\lambda; K_1)K_1 y$ for every $y \in D(K^2)$. Since $K_1 R(\lambda; K_1)$ is uniformly bounded, then K is closed and $D(K)$ is dense in X, and it follows that $KR(\lambda; K_1)y = R(\lambda; K_1)K_1 y$ for every $y \in X$ and $K \in \omega\text{-}ORCP_n$. This means that $D(K) \supset Range\ R(\lambda; K_1) = D(K_1)$ and $K \supset K_1$. Hence, $K = K_1$ and this complete the proof.

Theorem 3.3

Suppose $K \in \omega\text{-}OCP_n$ is the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$, let $f \in L^{'}(0, T; X)$ be continuous on $[0, T]$ and assume

$$v(t) = \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T. \tag{3.16}$$

System (1.2) has a solution x on $[0, T]$ for every $x \in D(K)$ if one of the following conditions is satisfied:

(i) $v(t)$ is continuously differentiable on $[0, T]$; and

(ii) $v(t) \in D(K)$ for $0 < t < T$ and $Kv(t)$ is continuous on $[0, T]$.

If (1.2) has a solution u on $[0, T]$ for some $x \in D(K)$ then v satisfies both (i) and (ii).

Proof:

Assume (1.2) has a solution u for some $x \in D(K)$, then this solution is given by (1.3). Consequently $v(t) = u(t) - T(t)x$ is differentiable for $t > 0$ as the difference of two such differentiable functions and $v^{'}(t) = u^{'}(t) - T(t)Kx$ is obviously continuous on $[0, T]$. Therefore (i) is satisfied. Also, if $x \in D(K)$, $T(t)x \in D(K)$ for $t \geq 0$, therefore $v(t) = u(t) - T(t)x \in D(K)$ for $t > 0$ and

$$Kv(t) = Ku(t) - KT(t)x = u^{'}(t) - f(t) - T(t)Kx$$

is continuous on $[0, T]$. Thus, (ii) is satisfied. On the other hand, it is easy to verify for $h > 0$, the identity

$$\frac{T(h) - I}{h} v(t) = \frac{v(t+h) - v(t)}{h} = -\frac{1}{h} \int_t^{t+h} T(t+h-s) f(s) ds. \tag{3.17}$$

From the continuity of f, it is clear that the second term on the right-hand of (3.17) has a limit $f(t)$ as $h \to 0$. Suppose $v(t)$ is continuously differentiable on $[0.T]$, then it follows from (3.17) that $v(t) \in D(K)$ for $0 < t < T$ so that

$$Kv(t) = v^{'}(t) - f(t), \tag{3.18}$$

since $v(0) = 0$, it implies that

$$u(t) = T(t) + v(t)$$

is the solution of (1.2) for $x \in D(K)$ and $A \in \omega\text{-}OCP_n$. Assume $v(t) \in D(K)$, then it follows from (3.17) that $v(t)$ is differentiable from the right at $t \geq 0$ and the right derivative $D^+ v(t)$ of v satisfies

$$D^+ v(t) = Kv(t) + f(t).$$

Since $D^+ v(t)$ is continuous, then $v(t)$ is continuously differentiable and

$$v^{'}(t) = Kv(t) + f(t).$$

if $v(0) = 0$, then $u(t) = T(t)x + v(t)$ is the solution of (1.2) for $x \in D(K)$ and the proof is complete.

Proposition 3.4

Let $K \in \omega\text{-}ORCP_n$ be the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$, suppose $f(s)$ is continuously differentiable on $[0, T]$, then :

(i) The Initial Value Problem (1.2) has a unique solution u on $[0,T]$ for every $x \in D(K)$. (ii) Assume $f \in L^{'}(0,T;X)$ be continuous on $[0,T]$ and $f(s) \in D(K)$ for $0 < s < T$ so that $Kf(s) \in L^{'}(0,T;X)$ for every $x \in D(K)$ and $K \in \omega$-$ORCP_n$, then system (1.2) has a unique solution on $[0,T]$.

Proof:

To prove (i), Assume

$$v(t) = \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds. \tag{3.19}$$

It is clear from (3.19) that $v(t)$ is differentiable for $t > 0$ and that its derivative

$$v^{'}(t) = T(t)f(0) + \int_0^t T(s)f(t-s)ds = T(t)f(0) + \int_0^t T(t-s)f^{'}(s)ds$$

is continuous on $[0,T]$. The results follow from Theorem 3.3 and this complete the proof of (i).

To prove (ii), from the conditions of the proposition, it follows that for $s > 0$, $A \in \omega$-$ORCP_n$, $T(t-s)f(s) \in D(K)$ and that $KT(t-s)f(s) = T(t-s)Kf(s)$ is integrable. Therefore $v(t)$ defined by (3.16) satisfies $v(t) \in D(K)$ for $t > 0$, $K \in \omega$-$ORCP_n$ and

$$Kv(t) = K\int_0^t T(t-s)f(s)ds = \int_0^t T(t-s)Kf(s)ds. \tag{3.20}$$

is continuous. Then, the result follows from Theorem 3.3 which completes the proof.

Theorem 3.5

Let $K \in \omega$-OCP_n be the infinitesimal generator of an analytic semigroup $\{T(t); t \geq 0\}$ and let $f \in L^p(0,T;X)$ with $1 < p < \infty$. Suppose u is the mild solution of (1.2), then u is Hölder Continuous with exponent $(p-1)/p$ on $[\epsilon, T]$ for every $\epsilon > 0$. Assume $x \in D(K)$, then u is Hölder Continuous with same exponent on $[0,T]$.

Proof:

Assume $\|T(t)\| \leq M$ on $[0,T]$. Since $T(t)$ is an analytic semigroup, then there is constant C such that $\|KT(t)\| \leq Ct^{-1}$ on $[0,T]$. This implies that $T(t)x$ is Lipschitz continuous on $[\epsilon,T]$ for any given $\epsilon > 0$. If $x \in D(K)$, $K \in \omega\text{-}OCP_n$, and $T(t)$ is Lipschitz continuous on $[0,T]$. It suffices therefore to show that if $f \in L^P(0,T;X)$, $1 < p < \infty$ then $v(t) = \int_0^t T(t-s)f(s)ds$ is Hölder Continuous with the same exponent $(p-1)/p$ on $[0,T]$. For $h > 0$, we have

$$v(t+h)-v(t) = \int_t^{t+h} T(t+h-s)f(s)ds + \int_0^t (T(t+h-s)-T(t-s))f(s)ds = I_1+I_2.$$

We estimate I_1 and I_2 separately. For I_1, we use Hölder's inequality to obtain,

$$\|I_1\| \leq M\int_t^{t+h} \|f(s)\|ds \leq Mh^{(p-1)/p}\Big(\int_t^{t+h} \|f(s)\|^p ds\Big)^{\frac{1}{p}} \leq M|f|_p h^{(p-1)/p,} \tag{3.21}$$

where $|f|_p = (\int_0^T \|f(s)\|^p ds)^{\frac{1}{p}}$ is the norm of f in $L^p(0,T;X)$. In order to estimate I_2, for $h > 0$, we have

$$\|T(t+h) - T(t)\| \leq 2M \quad for\ t, t+h \in [0,T]$$

and

$$\|T(t+h) - T(t)\| \leq C\frac{h}{t} \quad for\ t, t+h \in [0,T].$$

Therefore

$$\|T(t+h) - T(t)\| \leq C_1\mu(h,t) = C_1\ min\Big(1, \frac{h}{t}\Big) \quad for\ t, t+h \in [0,T], \tag{3.22}$$

where C_1 is a constant satisfying $C_1 \geq max(2M_1C)$. Using (3.22) and

Hölder's inequality, we have

$$\|I_2\| \le C_1 \int_0^t \mu(h,t-s)\|f(s)\|ds \le C_1|f|_p\Big(\int_0^t \mu(h,t-s)^{p/(p-1)}ds\Big)^{(p-1)/p}. \tag{3.23}$$

Since $\mu \ge 0$, we have

$$\int_0^t \mu(h,t-s)^{p/(p-1)}ds = \int_0^t \mu(h,\tau)^{p/(p-1)}d\tau \le \int_0^\infty \mu(h,\tau)^{p/(p-l)}d\tau = ph, \tag{3.24}$$

by combining (3.23) with (3.24), we find that

$$\|I_2\| \le Const.h^{(p-1)/p}.$$

Hence the proof of the theorem.

Theorem 3.6

Let $K \in \omega\text{-}ORCP_n$ be the infinitesimal generator of an analytic semigroup $\{T(t); t \ge 0\}$. Assume $f \in L^1(0,T;X)$ and suppose that for every $0 < t < T$, there exists a $\delta_1 > 0$ and a continuous real value function

$$\Omega_t(\tau) : [0,\infty] \to [0,\infty]$$

such that

$$\|f(t) - f(s)\| \le \Omega_t(|t-s|), \tag{3.25}$$

and

$$\int_0^{\delta_1} \frac{\Omega_t(\tau)}{\tau}d\tau < \infty. \tag{3.26}$$

Then, for every $x \in X$, the mild solution of (1.2) is a classical solution.

Proof:

Since $T(t)$ is an analytic semigroup, then $T(t)x$ is the solution of the homogeneous equation with initial data x for every $x \in X$. To show that the theorem is sufficient, by Theorem 3.3, we need to prove that

$$v(t) = \int_0^t T(t-s)f(s)ds \in D(K) \quad for\ 0 < t < T$$

and that $Kv(t)$ is continuous on this interval. To this end, we have

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= \int_0^t T(t-s)(f(s)-f(t))ds + \int_0^t T(t-s)f(t)ds. \end{aligned} \tag{3.27}$$

Suppose $v_2(t) \in D(K)$ and that $Kv_2(t) = (T(t) - I)f(t)$. By assumption of the theorem, its implies that f is continuous on $[0, T]$, it follows that $Kv_2(t)$ is continuous on $[0, T]$. To prove the same conclusion for v_1, we define

$$v_{1.\epsilon}(t) = \int_0^{t-\epsilon} T(t-s)(f(s)-f(t))ds \quad for\ t \geq \epsilon \tag{3.28}$$

and

$$v_{1.\epsilon}(t) = 0 \quad for\ t < \epsilon. \tag{3.29}$$

From (3.28) and (3.29), it is clear that $v_{1.\epsilon}(t) \to v_1(t)$ as $\epsilon \to 0$. It is also clear that $v_{1.\epsilon} \in D(K)$ and for $t \geq \epsilon$, we have

$$Kv_{1.\epsilon}(t) = \int_0^{t-\epsilon} KT(t-s)(f(s)-f(t))ds. \tag{3.30}$$

From (3.25) and (3.26), it follows that for $t > 0$, $Kv_{1.\epsilon}(t)$ converges as $\epsilon \to 0$ and that

$$\lim_{\epsilon \to 0} Kv_{1.\epsilon}(t) = \int_0^{t-\epsilon} KT(t-s)(f(s)-f(t))ds. \tag{3.31}$$

The closeness of K then implies that $v_1(t) \in D(K)$ for $t > 0$, we have

$$Kv_1(t) = \int_0^t KT(t-s)(f(s)-f(t))ds. \tag{3.32}$$

To conclude the proof, we have to show that $Kv_1(t)$ is continuous on $[0,T]$ for $0<\delta<t$, hence

$$Kv_1(t)=\int_0^\delta KT(t-s)(f(s)-f(t))ds+\int_\delta^t KT(t-s)(f(s)-f(t))ds. \quad (3.33)$$

For fixed $\delta>0$, the second integral on the right of (3.33) is a continuous function of t while the first integrals is of $O(\delta)$ uniformly in t. Thus, $K_1(t)$ is continuous and the proof is complete.

Theorem 3.7

Let $K\in\omega\text{-}OCP_n$ be the infinitesimal generator of an analytic semigroup $\{T(t);t\geq 0\}$ and suppose $f\in C^\zeta([0,T];X)$, if

$$v_1(t)=\int_0^t T(t-s)(f(s)-f(t))ds, \quad (3.34)$$

then $v_1\in D(K)$ for $0\leq t\leq T$ and $Kv_1(t)\in C^\zeta([0;T]:X)$.

Proof:

The fact that $v_1\in D(K)$ for $0\leq t\leq T$ is an immediate consequence of the proof of Theorem 3.6, hence, we only need to prove the Hölder Continuity of $Kv_1(t)$. Suppose that $\|T(t)\|\leq M$ on $[0,T]$ and that

$$\|KT(t)\|\leq Ct^{-1} \quad for \;\; 0<t\leq T. \quad (3.35)$$

Thus, for every $0<s<t\leq T$, we have

$$\begin{aligned}\|KT(t)-KT(s)\| &= \left\|\int_s^t K^2T(\tau)d\tau\right\|\leq\int_s^t\|K^2T(\tau)\|d\tau \\ &\leq 4C\int_s^t \tau^{-2}d\tau=4Ct^{-1}s^{-1}(t-s).\end{aligned} \quad (3.36)$$

Let $t \geq 0$ and $h > 0$, then

$$\begin{aligned} Kv_1(t+h) - Kv_1(t) = K\int_0^t (T(t+h-s) - T(t-s))(f(s) - f(t))ds \\ +K\int_0^t T(t+h-s)(f(t) - f(t+h))ds \\ +K\int_t^{t+h} T(t+h-s)(f(s) - f(t+h))ds \\ = I_1 + I_2 + I_3. \end{aligned} \tag{3.37}$$

We estimate each of the the three terms separately, first from (2.1) and (3.36),

$$\begin{aligned} \|I_1\| \leq \int_0^t \|KT(t+h-s) - KT(t-s)\|\|f(s) - f(t)\|ds \\ \leq 4CLh\int_0^t \frac{ds}{(t-s+h)(t-s)^{1-\zeta}} \leq C_1 h^{\zeta}. \end{aligned} \tag{3.38}$$

To estimate I_2, we refer to Definition 2.8 and Definition 2.9 so that

$$\begin{aligned} \|I_2\| = \|(T(t+h) - T(h))(f(t) - f(t+h))\| \\ \leq \|T(t+h) - T(h)\|\|f(t) - f(t+h)\| < 2MIh^{\zeta}. \end{aligned} \tag{3.39}$$

Finally, to estimate I_3, we use (3.35) and (2.1) to get

$$\begin{aligned} \|I_3\| \leq \int_t^{t+h} \|KT(t+h-s)\|\|f(s) - f(t+h)\|ds \\ \leq CL\int_t^{t+h} (t+h-s)^{\zeta-1}ds \leq C_2 h^{\zeta}. \end{aligned} \tag{3.40}$$

Combining (3.37) with estimates (3.38) and (3.39), we observe that $Kv_1(t)$ is Hölder continuous with exponent ζ on $[0, T]$. The proof is complete.

Theorem 3.8

Suppose $K \in \omega\text{-}ORCP_n$ is the infinitesimal generator of an analytic semigroup $\{T(t); t \geq 0\}$ and let $f \in C^{\zeta}([0, T]; X)$. If u is the solution of IVP

(1.3) on $[0,T]$ then:

(i) for every $\delta > 0$, $Ku \in C^{\zeta}([\delta,T]:X)$, hence $du/dt \in \zeta([\delta,T]:X)$;

(ii) if $x \in D(K)$, hence Ku and du/dt are continuous on $[0,T]$; and

(iii) if $x = 0$ and $f(0) = 0$, hence $T(t)f(t) \in C^{\zeta}([\delta,T]:X)$.

Proof:

Since

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds = T(t-s)f(s)ds = T(t)x + v(t). \quad (3.41)$$

and by (3.36), $KT(t)x$ is Lipschitz continuous on $\delta \le t \le T$, for every $\delta > 0$, it suffices to show that $Kv(t) \in C^{\zeta}([\delta,T]:X)$. To this end, we decompose v as

$$v(t) = v_1(t) + v_2(t) = \int_0^t T(t-s)(f(s) - f(t))ds + \int_0^t T(t-s)f(t)ds.$$

From Theorem 3.7, it follows that $Kv(t) \in C([0,T];X)$, it remains to show that $Kv_2(t) \in C^{\zeta}([\delta,T]:X)$ for $\delta > 0$. But $Kv_2(t) = (T(t)-I)f(t)$ and since $f \in C^{\zeta}([0,T];X)$, then we only have to show that $T(t)f(t) \in C^{\zeta}([\delta,T]:X)$ for every $\delta > 0$. Let $t \ge \delta$ and $h > 0$, then

$$\begin{aligned} &\|T(t+h)f(t+h) - T(t)f(t)\| \\ \le\ &\|T(t+h)\|\|f(t+h) - f(t)\| + \|T(t+h) - T(t)\|\|f(t)\| \\ \le\ &MLh^{\zeta} + \frac{C}{\delta}h\|f\|_{\infty} \le C_1 h^{\zeta}, \end{aligned} \quad (3.42)$$

here we used (3.22) and (2.1). Define

$$\|f\|_{\infty} = max_{0 \le t \le T}\|F(t)\|,$$

this complete the proof of (i). To prove (ii), we noted that if $x \in D(K)$, then $KT(t)x \in C([0,T];X)$. By Theorem 3.7, $Kv_1(t) \in C^{\zeta}([0,T];X)$ and since

f is continuous on $[0,T]$, it remains to show that $T(t)f(t)$ is continuous on $[0,T]$. From (i) it is clear that $T(t)f(t)$ is continuous on $[0,T]$. The continuity at $t=0$ follows readily from

$$\|T(t)f(t)-f(0)\| \leq \|T(t)f(0)-f(0)\| + M\|f(t)-f(0)\|$$

and the proof of (ii) is complete. Finally, to prove (iii), in the case $T(t)f(t) \in C^{\zeta}([0,T];X)$, it follows that

$$\begin{aligned} &\|T(t+h)f(t+h)-T(t)f(t)\| \\ &\leq \|T(t+h)\|\|f(t+h)-f(t)\| + \|T(t+h)-T(t)f(t)\| \\ &\leq MLh^{\zeta} + \|\int_t^{t+h} KT(\tau)f(t)d\tau\| \\ &\leq MLh^{\zeta} + \int_t^{t+h} \|KT(\tau)(f(t)-f(0))\|d\tau \\ &\leq MLh^{\zeta} + CL\int_t^{t+h} \tau^{-1}t^{\zeta}d\tau \leq MLh^{\zeta} + CL\int_t^{t+h} \tau^{\zeta-1}d\tau \leq Ch^{\zeta}. \end{aligned} \tag{3.43}$$

which complete the proof.

Theorem 3.9

Let X be a Banach space and for every t, $0 \leq t \leq T$, $A(t)$ is a bounded linear operator on X where $A \in \omega\text{-}ORCP_n$. If the function $t \to A(t)$ is continuous in the uniform operator then for every $x \in X$, the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) & 0 \leq s \leq t \leq T \\ u(s) = x \end{cases} \tag{3.44}$$

has a unique classical solution u.

Proof:

To proof this problem, we need to use the Picard's iterations method. Let

$\alpha = max_{0\leq t\leq T}\|A(t)\|$ and define a mapping S from $C([s,T]:X)$ into itself by

$$(Su)(t) = x + \int_s^t A(\tau)u(\tau)d\tau. \tag{3.45}$$

Denoting $\|u\|_\infty = max_{s\leq t\leq T}\|u(t)\|$, then it is easy to check that

$$\|Su(t) - Sv(t)\| \leq \alpha(t-s)\|u-v\|_\infty, \quad s \leq t \leq T. \tag{3.46}$$

Using (3.45) and (3.46), it follows by induction that

$$\|S^n u(t) - S^n v(t)\| \leq \frac{\alpha^n (t-s)^n}{n!}\|u-v\|_\infty, \quad s\leq t\leq T.$$

and therefore,

$$\|S^n u - S^n v\| \leq \frac{\alpha^n (T-S)^n}{n!}\|u-v\|_\infty.$$

For n large enough $\frac{\alpha^n(T-S)^n}{n!} < 1$ and by a well known generalization of the Banach contraction principle, S has a unique fixed point u in $C([s,T]:X)$ for which

$$u(t) = x + \int_s^t A(\tau)u(\tau)d\tau. \tag{3.47}$$

Since u is continuous, then the right hand side of (3.47) is differentiable. Thus u is differentiable and its derivative obtained by differentiating (3.47) satisfies $u'(t) = A(t)u(t)$. So u is a solution of the initial value problem (3.44). Since every solution of (3.44) is also a solution of (3.47), then the solution of (3.44) is unique and this complete the proof.

Theorem 3.10

Let the solution solution operator of the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) \qquad 0\leq s\leq t\leq T \\ u(s) = x \end{cases}$$

be defined by

$$u(t,s)x = u(t) \tag{3.48}$$

for every $0 \leq s \leq t \leq T$, $U(t,s)$ is a bounded linear operator and

(i) $\|u(t,s)\| \leq exp(\int_s^t \|A(\tau)\| d\tau) \forall A \in \omega\text{-}ORCP_n$;

(ii) $U(t,t) = I,\ U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$;

(iii) $(t,s) \to U(t,s)$ is continuous in the uniform operator topology for $0 \leq s \leq t \leq T$;

(iv) $\partial u(t,s)/\partial t = A(t)U(t,s)$ for $0 \leq s \leq t \leq T$; and

(v) $\partial u(t,s)/\partial s = -U(t,s)A(s)$ for $0 \leq s \leq t \leq T$.

Proof:

Since the problem (1.4) is linear, then it is obvious that $U(t,s)$ is an operator on all of X. From (3.47) its follows that

$$\|u(t)\| \leq \|x\| + \int_s^t \|A(\tau)\| \|u(\tau)\| d\tau$$

which by Gronwall's inequality implies

$$\|u(t,s)x\| = \|u(t)\| \leq \|x\| exp\Big(\int_s^t \|A(\tau)\| d\tau\Big) \tag{3.49}$$

and so $U(t,s)$ is bounded and satisfies (i). From (3.48) it follows readily that $U(t,t) = I$ and from the uniqueness of the solution of (1.4), the relation $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$, follows by combining (i) and (ii), then (iii) follows. Finally from (3.47) and (iii), then it follows that $U(t,s)$ is the unique solution of the integral equation

$$U(t,s) = I + \int_s^t A(\tau)U(\tau,s)d\tau \tag{3.50}$$

in $B(X)$(space of all bounded linear operator on X). Differentiating (3.50) with respect to t yields (iv). Differentiating (3.50) with respect to s, we get

$$\frac{\partial}{\partial s}U(t,s) = -A(s) + \int_s^t A(\tau)\frac{\partial}{\partial s}U(\tau,s)d\tau. \tag{3.51}$$

From the uniqueness of the solution of (3.50), it follows that

$$\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s) \tag{3.52}$$

and this complete the proof.

Theorem 3.11

Let $A(t) \in \omega\text{-}ORCP_n$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on the Banach space X for $t \in [0,T]$. The family family of generators $(A(t))_{t\in[0,T]}$ is stable if and only if there are constants $M \geq 1$ and ω such that $\rho(A(t)) \supset (\omega, \infty)$ for $t \in [0,T]$ and either one of the following conditions is satisfied:

$$(i)\Big\|\prod_{i=1}^{k} T_{t_i(s_i)}\Big\| \leq Mexp\Big\{\omega\sum_{i=1}^{k} s_i\Big\} \quad for \quad s_i \geq 0 \tag{3.53}$$

and any finite sequence $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq T$, $k = 1, 2, ...$

or

$$(ii)\Big\|\prod_{i=1}^{k} R(\lambda_i; A(t_i))\Big\| \leq M\prod_{i=1}^{k}(\lambda_i - \omega)^{-1} \quad for \quad \lambda_i > \omega \tag{3.54}$$

and any finite sequence $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq T$, $k = 1, 2, ...$

Proof:

From the statement of the theorem, it is clear that its suffices to prove that for a family $(A(t))_{t\in[0,T]}$ of infinitesimal generators for which $\rho(A(t)) \supset (\omega, \infty)$, the estimates (2.2), (3.10) and (3.11) are equivalent. Assume (2.2) holds and s_i, $1 \leq i \leq k$ be positive rational numbers. Let $\lambda = N$ be positive integer such that $Ns_i = m_i$ is a positive integer for $1 \le i \le k$. In (2.2), we take $m = \sum_{i=1}^{k} m_i$ terms and subdivided them into k subsets containing m_i, $1 \le i \le k$, terms. All values of t in the $i-th$ subset are taken to be equal to t_i. Dividing both sides of inequality by N^m we get

$$\Big\| \prod_{i=1}^{k} [\frac{m_i}{s_i} R(\frac{m_i}{s_i}; A(t_i))]^{m_i} \Big\| \le M\Big(1 - \frac{\omega}{N}\Big)^{-m}. \tag{3.55}$$

Letting $N \to \infty$, such that Ns_i, $1 \le i \le k$, and suppose each of the integers of m_i tends to infinity and by the exponential formula, then we obtain

$$\Big\| \prod_{i=1}^{k} T_i(s_i) \Big\| \le Mexp\Big\{\omega \sum_{i=1}^{k} s_i\Big\}$$

and that proved (i) for all positive rationals s_i. To prove (ii), let consider the general case of non-negative real s_i follows from the strong continuity of $T_t(s)$ and thus (2.2) implies (3.53). Assume

$$R(\lambda_i; A(t_i))x = \int_0^\infty e^{-\lambda_i s} T_t(s) x ds \quad for \quad \lambda_i > \omega \tag{3.56}$$

Iterating (3.56) a finite number of times yields

$$\prod_{i=1}^{k} R(\lambda_i; A(t_i))x = \int_0^\infty ... \int_0^\infty exp\Big\{ - \sum_{i=1}^{k} \lambda_i s_i \Big\} \prod_{i=1}^{k} T_t(s_i) x ds_1 ... ds_k. \tag{3.57}$$

Using (3.53) to estimates the norm of the right-hand side of (3.57), we have

$$\Big\| \prod_{i=1}^{k} R(\lambda_i; A(t_i))x \Big\| \le M\|x\| \prod_{i=1}^{k} \int_0^\infty e^{(\omega - \lambda_i)s_i} ds_i = M\|x\| \prod_{i=1}^{k} (\lambda_i - \omega)^{-1}$$

and therefore (3.53) implies (3.54). Finally, choosing all λ_i equal to λ, then (3.54) shows that (3.55) implies (1.14) and this complete the proof.

Conclusion

The study presented some properties of $w-$order preserving and $w-$order reversing partial contraction mapping in semigroup of linear operators.It have been established that ω-order reversing partial contraction mapping (ω-$ORCP_n$) is a semigroup of linear operator that generates an evolution systems.

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