Fractional Orlicz-Sobolev spaces

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8^{th} European Congress of Mathematics
Minisymposium "Geometric-functional inequalities and related topics"
Portorož, June 22nd, 2021

- A.Alberico, A.C., L.Pick & L.Slavikóvá Fractional Orlicz-Sobolev embeddings, J. Math. Pures. Appl. (2021)
- A.Alberico, A.C., L.Pick & L.Slavikóvá On the limit as $s \to 0^+$ of fractional Orlicz-Sobolev spaces, J. Fourier Analysis and Appl. (2020)
- A.Alberico, A.C., L.Pick & L.Slavikóvá On the limit as $s \to 1^-$ of possibly non-separable fractional Orlicz-Sobolev spaces, Atti Accad. Naz. Lincei, Rend. Lincei Mat. Appl. (2020)

Integer order Orlicz-Sobolev spaces

The Orlicz-Sobolev spaces generalize the classical Sobolev space

$$W^{1,p}(\Omega) = \{u \,\text{weakly diff.} : \, u \in L^p(\Omega), \nabla u \in L^p(\Omega)\},$$

and its homogeneous counterpart

$$V^{1,p}(\Omega) = \{u \,\text{weakly diff.} : \, \nabla u \in L^p(\Omega)\}.$$

Here, Ω is an open set in $\mathbb{R}^n$.
The role of the Lebesgue space $L^p(\Omega)$ is played by a more general Orlicz space $L^A(\Omega)$.
In the definition of $L^A(\Omega)$, the power t^p is replaced by a general Young function, namely a convex function $A : [0, \infty) \to [0, \infty]$ such that $A(0) = 0$.

The Orlicz space $L^A(\Omega)$ is equipped with the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right)dx \le 1\right\}.$$

The Orlicz-Sobolev space is then defined as

$$W^{1,A}(\Omega) = \{u\,\text{weakly diff.} :\ u \in L^A(\Omega), \nabla u \in L^A(\Omega)\},$$

and its homogeneous version as

$$V^{1,A}(\Omega) = \{u\,\text{weakly diff.} :\ \nabla u \in L^A(\Omega)\}.$$

The Orlicz-Sobolev spaces are of crucial use in the analysis of Partial Differential Equations governed by nonlinearities which are not of polynomial type.
The class of Orlicz-Sobolev spaces is also rich enough to shed light on some aspects of the theory of classical Sobolev spaces.

This is apparent when Sobolev embeddings are in question.
The class of Lebesgue norms is not large enough to be closed under the operation of associating an optimal target in Sobolev embeddings.
Assume, for instance, that Ω is bounded and regular. Then

$$W^{1,p}(\Omega) \to \begin{cases} L^{\frac{np}{n-p}}(\Omega) & 1 \leq p < n \\ L^q(\Omega) \quad \forall\, q < \infty & p = n \\ L^\infty(\Omega) & p > n. \end{cases}$$

An optimal Lebesgue target space does not exist for $p = n$.
This drawback does not affect the broader class of Orlicz-Sobolev spaces.

Given any Young function A, there always exists another Young function A_n such that

$$W^{1,A}(\Omega) \to L^{A_n}(\Omega), \tag{1}$$

and $L^{A_n}(\Omega)$ is the optimal (smallest) Orlicz target space.
The function A_n is given by

$$A_n(t) = A(H_n^{-1}(t)) \quad \text{for } t > 0,$$

where $H_n : [0, \infty) \to [0, \infty)$ is defined as

$$H_n(t) = \left(\int_0^t \left(\frac{s}{A(s)} \right)^{\frac{1}{n-1}} ds \right)^{\frac{1}{n'}} \quad \text{for } t > 0$$

[C., Indiana Univ. Math. J. 1996], [C., Comm. Part. Diff. Eq. 1997].
Versions of embedding (1) for the spaces $W_0^{1,A}(\Omega)$ and $W^{1,A}(\mathbb{R}^n)$ are also available.

In particular, an application of this result yields:

$$W^{1,p}(\Omega) \to \begin{cases} L^{\frac{np}{n-p}}(\Omega) & 1 \leq p < n \\ \exp L^{n'}(\Omega) & p = n \\ L^{\infty}(\Omega) & p > n, \end{cases}$$

and tells us that all target spaces are optimal in the class of Orlicz spaces.

This recovers the Sobolev embedding for $p \neq n$, and the borderline Pohozaev-Trudinger-Yudovich embedding in the borderline case $p = n$.

It also informs us about their optimality.

Fractional order spaces

Various definitions of fractional order Sobolev spaces are available in the literature, including Besov, Lizorkin-Triebel, Bessel-potential spaces.
We focus on Gagliardo-Slobodeckij type spaces.
They have been the object of renewed interest in the last two decades, starting with the work of such authors as Bourgain, Brezis, Caffarelli, Maz'ya.
Classical setting. Let $s \in (0,1)$, $p \in [1,\infty)$, $\Omega = \mathbb{R}^n$.
The Gagliardo-Slobodeckij seminorm is given by

$$|u|_{s,p,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s} \right)^p \frac{dx\, dy}{|x-y|^n} \right)^{\frac{1}{p}}$$

The homogenous fractional space is defined as

$$V^{s,p}(\mathbb{R}^n) = \{u : |u|_{s,p,\mathbb{R}^n} < \infty\}, \qquad (2)$$

Classical fractional Sobolev embedding.

Let $1 \le p < \frac{n}{s}$. Then $\exists\, C$ s.t.

$$\|u\|_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \le C\,|u|_{s,p,\mathbb{R}^n}$$

for every measurable u decaying to 0 near infinity.

Although

$$\text{“}V^{1,p}(\mathbb{R}^n) \neq V^{1,p}(\mathbb{R}^n)\text{”},$$

in the sense that setting $s = 1$ in the definition of fractional order space dose not recover the integer order one, the Sobolev exponent $\frac{np}{n-p}$ is reproduced on setting $s = 1$ in $\frac{np}{n-sp}$.

Fractional order Orlicz-Sobolev spaces.

Given $s \in (0,1)$ and a Young function A, the fractional Orlicz-Sobolev seminorm is defined as

$$|u|_{s,A,\mathbb{R}^n} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dx\,dy}{|x-y|^n} \leq 1 \right\}.$$

The homogenoeus fractional Orlicz-Sobolev space $V^{s,A}(\mathbb{R}^n)$ is then defined as

$$V^{s,A}(\mathbb{R}^n) = \{u : |u|_{s,A,\mathbb{R}^n} < \infty\},$$

We deal with functions in $V^{s,A}(\mathbb{R}^n)$ that "decay to zero" near infinity. Precisely, functions in the space

$$V^{s,A}_d(\mathbb{R}^n) = \{u \in V^{s,A}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \ \forall t > 0\}.$$

Pb.: Optimal embeddings of $V^{s,A}_d(\mathbb{R}^n)$ with target space in classes of Banach function spaces.

Embeddings into Orlicz spaces.

Like in the integer order case, there exists an optimal Orlicz target space for embeddings of $V_d^{s,A}(\mathbb{R}^n)$.
Let A be a Young function such that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty, \tag{3}$$

and

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{4}$$

Assumption (3) amounts to requiring that A has a non-supercritical growth near infinity.
If $A(t) = t^p$, condition (3) corresponds to

$$1 \leq p \leq \frac{n}{s}\,.$$

Define the function $H : [0, \infty) \to [0, \infty)$ as

$$H_{\frac{n}{s}}(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \quad \text{for } t \geq 0$$

and the Young function $A_{\frac{n}{s}}$ by

$$A_{\frac{n}{s}}(t) = A(H_{\frac{n}{s}}^{-1}(t)) \quad \text{for } t \geq 0.$$

- The function $A_{\frac{n}{s}}$ is the optimal fractional Sobolev conjugate of A.

Note that setting $s = 1$ in $A_{\frac{n}{s}}$ recovers the optimal integer order Sobolev conjugate of A, although

$$\text{“}V^{1,A}(\mathbb{R}^n) \neq V^{1,A}(\mathbb{R}^n)\text{”}.$$

Theorem 1: Optimal fractional Orlicz target space

Let $s \in (0,1)$. Let A be a Young function such that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty \qquad \int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \qquad (5)$$

Then,

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n), \qquad (6)$$

and $\exists\, C$ s.t.

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \qquad \forall\, u \in V_d^{s,A}(\mathbb{R}^n). \qquad (7)$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal (smallest) target space in (6)–(7) among all Orlicz spaces.

Remark: The second condition in (5), is necessary for an embedding $V_d^{s,A}(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$, for any Young function B.

Example 1. Let A be a Young function such that

$$A(t) \approx t^p \quad \text{near infinity,}$$

with $1 \leq p \leq \frac{n}{s}$. Then Theorem 1 yields

$$V_d^{s,p}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n),$$

where

$$A_{\frac{n}{s}}(t) \approx \begin{cases} t^{\frac{np}{n-sp}} & \text{if } 1 \leq p < \frac{n}{s} \\ e^{t^{\frac{n}{n-s}}} & \text{if } p = \frac{n}{s} \end{cases} \quad \text{as } t \to \infty.$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal Orlicz target space.
This recovers the classical fractional Sobolev embedding $(1 \leq p < \frac{n}{s})$, and provides us with a Pohozaev-Trudinger-Yudovich type result $(p = \frac{n}{s})$.

Example 2. Let A be a Young function such that

$$A(t) \approx t^p (\log t)^\alpha \quad \text{as } t \to \infty,$$

where either $p = 1$ and $\alpha \geq 0$, or $1 < p < \frac{n}{s}$ and $\alpha \in \mathbb{R}$, or $p = \frac{n}{s}$ and $\alpha \leq \frac{n}{s} - 1$. Then, Theorem 1 tells us that

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n),$$

where

$$A_{\frac{n}{s}}(t) \approx \begin{cases} t^{\frac{np}{n-sp}} (\log t)^{\frac{\alpha n}{n-sp}} & \text{if } 1 \leq p < \frac{n}{s} \\ e^{t^{\frac{n}{n-(\alpha+1)s}}} & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\ e^{e^{t^{\frac{n}{n-s}}}} & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1 \end{cases} \quad \text{as } t \to \infty.$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal Orlicz target space.

This yields, in particular, embeddings for $V_d^{s,A}(\mathbb{R}^n)$ in the spirit of [Fusco-Lions-Sbordone, PAMS 1996] and [Edmunds-Gurka-Opic, IUMJ 1995].

Improvement: optimal rearrangement-invariant target space.

Back to classical fractional Sobolev spaces.
Let $s \in (0,1)$. If $1 \le p < \frac{n}{s}$, then

$$V_d^{s,p}(\mathbb{R}^n) \to L^{\frac{np}{n-sp},p}(\mathbb{R}^n). \tag{8}$$

[Frank-Seiringer, JFA 2008].
Here, $L^{\frac{np}{n-sp},p}(\mathbb{R}^n)$ is the Lorentz space equipped with the norm

$$\|u\|_{L^{\frac{np}{n-sp},p}(\mathbb{R}^n)} = \left\|r^{-\frac{s}{n}}u^*(r)\right\|_{L^p(0,\infty)}.$$

Since

$$L^{\frac{np}{n-sp},p}(\mathbb{R}^n) \subsetneq L^{\frac{np}{n-sp}}(\mathbb{R}^n),$$

embedding (8) actually improves the classical fractional Sobolev embedding.

Problem: improvement in a similar spirit for $V_d^{s,A}(\mathbb{R}^n)$.

Specifically, we seek the optimal rearrangement-invariant target space $Y(\mathbb{R}^n)$ for

$$V_d^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n).$$

Recall that a Banach function space $X(\mathbb{R}^n)$ is called a rearrangement-invariant space if

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \qquad \text{if} \quad u^* = v^* .$$

Let $s \in (0,1)$. Let A be a Young function as above, and let $a : [0,\infty) \to [0,\infty)$ be such that

$$A(t) = \int_0^t a(r)\,dr \qquad \text{for } t \geq 0.$$

Define the Young function $\widehat{A}$ as

$$\widehat{A}(t) = \int_0^t \widehat{a}(r)\,dr \quad \text{for } t \geq 0, \tag{9}$$

where

$$\widehat{a}^{\,-1}(r) = \left(\int_{a^{-1}(r)}^{\infty} \left(\int_0^t \left(\frac{1}{a(\varrho)} \right)^{\frac{s}{n-s}} d\varrho \right)^{-\frac{n}{s}} \frac{dt}{a(t)^{\frac{n}{n-s}}} \right)^{\frac{s}{s-n}} \qquad \text{for } r \geq 0.$$

One has that

$$\widehat{A}(t) \lesssim A(t) \quad \text{for } t \geq 0.$$

Moreover,

$$\widehat{A}(t) \approx A(t) \qquad \text{if} \quad A(t) << t^{\frac{n}{s}}$$

in the sense that the Matuszewska-Orlicz index $I(A) < \frac{n}{s}$, where

$$I(A) = \lim_{\lambda\to\infty} \frac{\log\left(\sup_{t>0} \frac{A(\lambda t)}{A(t)}\right)}{\log \lambda}.$$

- If $A(t) = t^p$ and $1 \leq p < \frac{n}{s} \implies \widehat{A}(t) \approx t^p$.
- If $A(t) \approx t^{\frac{n}{s}}$ as $t \to \infty \implies \widehat{A}(t) \approx \left(\frac{t}{\log t}\right)^{\frac{n}{s}}$ as $t \to \infty$.

Let $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ be the Orlicz-Lorentz space equipped with the norm

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} = \|r^{-\frac{s}{n}} u^*(r)\|_{L^{\widehat{A}}(0,\infty)} .$$

Theorem 2: Optimal r.i. target space

Let $s \in (0,1)$. Let A and $\widehat{A}$ be as above.
Then,

$$V_d^{s,A}(\mathbb{R}^n) \to L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n), \tag{10}$$

and $\exists\, C$ s.t.

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n} \tag{11}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$.
Moreover, $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ is the optimal r.i. target space in (10)–(11).

- Theorem 1 (optimal Orlicz target) is deduced from Theorem 2 (optimal r.i. target).
- Theorem 2 is in turn a consequence of a Hardy type inequality for functions in $V_d^{s,A}(\mathbb{R}^n)$.

Classical Hardy inequality: if $1 \le p < n$, then

$$\left\| \frac{u(x)}{|x|} \right\|_{L^p(\mathbb{R}^n)} \le C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for every $u \in V_d^{1,p}(\mathbb{R}^n)$.

A fractional Hardy inequality in $V_d^{s,p}(\mathbb{R}^n)$, for $1 \le p < \frac{n}{s}$, yields

$$\left\| \frac{u(x)}{|x|^s} \right\|_{L^p(\mathbb{R}^n)} \le C\, |u|_{s,p,\mathbb{R}^n}$$

for every $u \in V_d^{s,p}(\mathbb{R}^n)$ [Maz'ya-Shaposhnikova, JFA 2002] .

Theorem 3: Fractional Orlicz-Hardy inequality

Let $s \in (0,1)$. Let A and $\widehat{A}$ be as above. Then, there exists a constant C s.t.

$$\left\| \frac{u(x)}{|x|^s} \right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \leq C\, |u|_{s,A,\mathbb{R}^n}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{|x|^s} \right) dx \leq (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(C \frac{|u(x)-u(y)|}{|x-y|^s} \right) \frac{dx\, dy}{|x-y|^n}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$.

Example 3. Let A be a Young function such that

$$A(t) \approx t^p (\log t)^\alpha \quad \text{as } t \to \infty.$$

Then

$$\left\| \frac{u(x)}{|x|^s} \right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \leq C \, |u|_{s,A,\mathbb{R}^n}$$

where

$$\widehat{A}(t) \approx \begin{cases} t^p (\log t)^\alpha & \text{if } 1 \leq p < \frac{n}{s} \\ t^{\frac{n}{s}} (\log t)^{\alpha - \frac{n}{s}} & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\ t^{\frac{n}{s}} (\log t)^{-1} (\log(\log t))^{-\frac{n}{s}} & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1 \end{cases} \quad \text{as } t \to \infty.$$

Additional results on fractional Orlicz-Sobolev embeddings

- Embeddings for higher-order fractional Orlicz-Sobolev spaces.
- Fractional Orlicz-Sobolev spaces defined on open bounded sets $\Omega \subseteq \mathbb{R}^n$.
- Compact embeddings.

Limits of normalized seminorms as $s \to 0^+$ and $s \to 1^-$.

Recall that, setting $s = 0$ or $s = 1$ in the definition of the fractional space $V^{s,p}(\mathbb{R}^n)$, does not reproduce the space $L^p(\mathbb{R}^n)$ or $V^{1,p}(\mathbb{R}^n)$.

However, a result from [Maz'ya-Shaposhnikova (2002)] tells us that, if $p \geq 1$ and $u \in \cup_{s\in(0,1)} V^{s,p}_d(\mathbb{R}^n)$, then,

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)^p \frac{dx\,dy}{|x-y|^n} = \frac{2\,n\omega_n}{p}\int_{\mathbb{R}^n} |u(x)|^p\,dx.$$

Pb: What about the limit of $V^{s,A}(\mathbb{R}^n)$ as $s \to 0^+$?

Define the Young function $\overline{A}$ associated with A as

$$\overline{A}(t) = \int_0^t \frac{A(\tau)}{\tau}\, d\tau \qquad \text{for } t \geq 0\,.$$

One has that

$$A \approx \overline{A} \qquad \text{since} \qquad A(t/2) \leq \overline{A}(t) \leq A(t) \qquad \text{for } t \geq 0.$$

Recall that $A \in \Delta_2$ if $\exists C > 0$ s.t.

$$A(2t) \le CA(t) \qquad \text{for } t \ge 0.$$

For instance, if $\gamma > 0$ and

$$A(t) \approx e^{t^\gamma} \quad \text{as } t \to \infty \quad \text{and/or} \quad A(t) \approx e^{-t^{-\gamma}} \quad \text{as } t \to 0,$$

then $A \notin \Delta_2$.

Theorem 5: limit of $V_d^{s,A}(\mathbb{R}^n)$ as $s \to 0^+$

Let $A \in \Delta_2$. Assume that $u \in \bigcup_{s\in(0,1)} V_d^{s,A}(\mathbb{R}^n)$. Then

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} = 2n\omega_n \int_{\mathbb{R}^n} \overline{A}(|u(x)|)\,dx.$$

Note that Theorem 5 recovers [Maz'ya-Shaposhnikova (2002)] when $A(t) = t^p$ for some $p \ge 1$, since $\overline{A}(t) = \frac{1}{p}t^p$ in this case.

A partial result in this connection had been establsihed in [Capolli-Maione-Salort-Vecchi, 2019], where just estimates for $\liminf_{s\to 0^+}$ and $\limsup_{s\to 0^+}$ are proved, and additional assumptions on A are required.

Theorem 5 provides a full answer to the relevant problem. Indeed, the result can fail if the Δ_2-condition is dropped.

Theorem 6

There exist Young functions $A \notin \Delta_2$, and corresponding functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in V_d^{s,A}(\mathbb{R}^n)$ for every $s \in (0,1)$,

$$\int_{\mathbb{R}^n} \overline{A}(|u(x)|)\, dx \leq \int_{\mathbb{R}^n} A(|u(x)|)\, dx < \infty\,,$$

but

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^n} = \infty\,.$$

The analysis of the limit as $s \to 1^-$ was initiated by [Bourgain-Brezis-Mironescu (2001)].
A result from that paper tells us that if $1 \leq p < \infty$ and

$$u \in W^{1,p}(\mathbb{R}^n)\,,$$

then

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s} \right)^p \frac{dx\,dy}{|x-y|^n} = K(p,n) \int_{\mathbb{R}^n} |\nabla u|^p \, dx,$$

where

$$K(p,n) = \frac{1}{p} \int_{\mathbb{S}^{n-1}} |\theta \cdot e|^p \, d\mathcal{H}^{n-1}(\theta)\,,$$

- $\mathcal{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$;
- $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure;
- e is any point on $\mathbb{S}^{n-1}$.

A converse of this result is also shown in the paper [Bourgain-Brezis-Mironescu (2001)], which holds if $1 < p < \infty$.
Assume that $u \in L^p(\mathbb{R}^n)$. If

$$\liminf_{s\to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s} \right)^p \frac{dx\, dy}{|x-y|^n} < \infty\,,$$

then

$$u \in W^{1,p}(\mathbb{R}^n).$$

If $p = 1$, then the conclusion can fail!

A version holds in the space $BV(\mathbb{R}^n)$ of functions of bouned variation.

Pb: What about the limit of $W^{s,A}(\mathbb{R}^n)$ as $s \to 1^-$?

Define the Young function $A_\circ$ as

$$A_\circ(t) = \int_0^t \int_{\mathbb{S}^{n-1}} A(r\,|\theta \cdot e|)\, d\mathcal{H}^{n-1}(\theta)\, \frac{dr}{r} \qquad \text{for } t \geq 0\,, \tag{12}$$

where e is any fixed vector in $\mathbb{S}^{n-1}$.

- The right-hand side of (12) is independent of the choice of e.
- $A_\circ$ is always equivalent to A, namely $\exists\, c_1, c_2$ s.t.

$$A(c_1 t) \leq A_\circ(t) \leq c_2 A(t) \quad \text{for } t \geq 0.$$

Theorem 7: limit of $W^{s,A}(\mathbb{R}^n)$ as $s \to 1^+$

Let A be a finite-valued Young function. If

$$u \in W^{1,A}(\mathbb{R}^n),$$

then, there exists $\lambda_0 > 0$ such that

$$\lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s}\right) \frac{dx\, dy}{|x-y|^n} = \int_{\mathbb{R}^n} A_\circ\left(\frac{|\nabla u|}{\lambda}\right) dx$$

for every $\lambda \geq \lambda_0$.

In the light of the restriction $p > 1$ in the "converse" result by [Bourgain-Brezis-Mironescu] to imply that $u \in W^{1,p}(\mathbb{R}^n)$, a converse to our Theorem 7 requires some additional assumptions on A, which rule out the case where $A(t) = t$. They amount to:

$$\lim_{t\to\infty} \frac{A(t)}{t} = \infty, \quad \text{superlinear growth near infinity} \tag{13}$$

and

$$\lim_{t\to 0^+} \frac{A(t)}{t} = 0 \quad \text{sublinear decay at } 0. \tag{14}$$

- If $A(t) = t^p$, then conditions (13) and (14) correspond to requiring that

$$p > 1\,.$$

A Young function A is called N-**function** if it is finite-valued, strictly positive and fulfils conditions (13) and (14).

A converse to Theorem 7 holds for the subclass of N- functions A.

Theorem 8

Let A be an N-function. If $u \in L^A(\mathbb{R}^n)$ and $\exists\, \lambda > 0$ s.t.

$$\liminf_{s \to 1^-} \, (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dx\, dy}{|x-y|^n} < \infty,$$

then $u \in W^{1,A}(\mathbb{R}^n)$.

- If A has a linear growth near infinity or near zero, then a counterpart of theses results holds for the relaxed functional of

$$\int_{\mathbb{R}^n} A(|\nabla u|)\, dx$$

in the space space $BV(\mathbb{R}^n)$ of functions of bounded variation.