Closed G_2-structures on compact locally homogeneous spaces

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G_2-structures

Definition

A G_2-structure on a 7-manifold M is given by a 3-form φ with pointwise stabilizer isomorphic to G_2.

- Pointwise $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$.
- φ is non-degenerate: $i_X\varphi \wedge i_X\varphi \wedge \varphi \neq 0$, for every $X \neq 0$.

$\rightsquigarrow$ φ induces a Riemannian metric g_φ with volume form dV_φ:

$$g_\varphi(X, Y)dV_\varphi = \frac{1}{6} i_X\varphi \wedge i_Y\varphi \wedge \varphi.$$

Proposition (Fernández-Gray)

The following are equivalent:

(a) $\nabla^{LC}\varphi = 0$;

(b) $d\varphi = 0$ and $d(*\varphi) = 0$;

(c) $Hol(g_\varphi)$ is isomorphic to a subgroup of G_2.

A G_2-structure satisfying $(a), (b)$ or (c) is called parallel.

Remark

- The conditions $\nabla^{LC}\varphi = 0$ and $d(*\varphi) = 0$ are non-linear in φ.
- Metrics induced by parallel G_2-structures are Ricci-flat [Bonan].

A G_2-structure φ is closed (or calibrated) if $d\varphi = 0$. Then

$$d * \varphi = \tau \wedge \varphi,$$

where $\tau \in \Lambda^2_{14} \cong \mathfrak{g}_2$, i.e. $\tau \wedge \varphi = - * \tau$ and $\tau \wedge *\varphi = 0$.

Remark

- $\tau = d^*\varphi \Rightarrow d^*\tau = 0 \Rightarrow d\tau = \Delta_\varphi \varphi$, where $\Delta_\varphi = dd^* + d^*d$ is the Hodge Laplacian.
- φ defines a calibration on M (i.e. $\varphi|_\xi \leq vol_\xi$, $\forall$ tg oriented 3-plane ξ) [Harvey-Lawson].
- $Scal(g_\varphi) = -\frac{1}{2}|\tau|^2 \leq 0$ [Bryant] $\rightsquigarrow$ no restrictions on compact manifolds!

Automorphism group

Remark

General results on the existence of closed G_2-structures on (compact) 7-manifolds are still not known.

$Aut(M,\varphi) := \{f \in Diff(M) | f^*\varphi = \varphi\} \Rightarrow$ when M is compact its Lie algebra is $\mathfrak{aut}(M,\varphi) = \{X \in \chi(M) | L_X\varphi = 0\}$.

Theorem (Podestá-Raffero)

M compact with φ closed non-parallel. Then

- $\dim \mathfrak{aut}(M,\varphi) \leq b_2(M)$;
- $\mathfrak{aut}(M,\varphi)$ *is abelian with dim* ≤ 6.

$\hookrightarrow$ There are no compact homogeneous examples with invariant (non-parallel) closed G_2-structures.

Compact locally homogeneous examples

The first known example of compact manifold admitting a closed G_2-structure but with no parallel G_2-structures is a nilmanifold $\Gamma\backslash N$ [Fernández].

Problem

Study the existence of invariant closed G_2-structures on compact locally homogeneous $\Gamma\backslash G$, with G Lie group.

$\Gamma\backslash G$ with an invariant closed G_2-structure $\varphi \longleftrightarrow (\mathfrak{g}, \varphi)$

Remark

$\mathfrak{g}$ has to be unimodular, i.e. $\mathrm{tr}(\mathrm{ad}_X) = 0$, for every $X \in \mathfrak{g}$.

Classification results on Lie algebras

- Unimodular non-solvable Lie algebras [F-Raffero]

$\hookrightarrow$ $\mathfrak{g}$ must have Levi decomposition $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{r}$ with

$\mathfrak{r}$ centerless if $\mathfrak{g}$ is a product (3 classes of isomorphism)

$\mathfrak{r} \cong \mathbb{R} \ltimes \mathbb{R}^3$ if $\mathfrak{g}$ is not a product (1 class of isomorphism).

$\hookrightarrow$ A unimodular Lie algebra with non-trivial center admitting a closed G_2-structure must be solvable!

Problem

Classify all unimodular Lie algebras with non-trivial center admitting closed G_2-structures, up to isomorphism.

- In the nilpotent case the are 12 classes of isomorphisms [Conti-Fernández].

Non-trivial center

Theorem (F-Raffero-Salvatore)

- *There exist 11 isomorphism classes of unimodular non-nilpotent Lie algebras with non-trivial center admitting a closed G_2-structure.*
- *Two of the isomorphism classes are the contactization of a symplectic Lie algebra.*

For the proof we use that $\mathfrak{g}$ has to be the central extension of a symplectic Lie algebra $\mathfrak{h}$ endowed with a closed (possibly non-degenerate) 2-form ω_0, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$, with

$$[z, \mathfrak{h}] = 0, \ [x, y] = -\omega_0(x, y)z + [x, y]_{\mathfrak{h}}, \ \forall x, y \in \mathfrak{h}.$$

Laplacian flow

Idea: use a geometric flow to deform a closed G_2-structure and eventually obtain a parallel one

Definition (Bryant)

Let φ_0 be a a closed G_2-structure on M^7. The Laplacian flow (LF) is

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0. \end{cases}$$

where $\Delta_{\varphi(t)}$ is the Hodge Laplacian of $g_{\varphi(t)}$.

if $\varphi(t)$ solves the LF, then $\varphi(t) \in [\varphi_0] \in H^3_{DR}(M^7)$ and

$$\partial_t g_{\varphi(t)} = -2Ric(g_{\varphi(t)}) + l.o.t.$$

Remark

If M^7 is compact, then

- stationary points are parallel G_2-structures.
- the LF is the gradient flow of Hitchin's volume functional $\mathcal{V} : \varphi \in [\varphi_0] \mapsto \int_M \varphi \wedge *\varphi$.

$\mathcal{V}$ is monotonically increasing along the LF, its critical points are parallel G_2-structures and they are strict local maxima.

Theorem (Bryant-Xu)

Assume that (M^7, φ_0) is compact. Then the LF has a unique solution for short time $t \in [0, \epsilon)$, with ϵ depending on $\varphi_0 = \varphi(0)$.

Recent developments

- If φ_0 is near a torsion-free G_2-structure $\tilde{\varphi}$, then the LF converges to a torsion-free G_2-structure which is related to $\tilde{\varphi}$ via a diffeomorphism [Xu-Ye; Lotay-Wei].

- Shi-type derivative estimates for Rm and τ along the flow: a bound on $\Lambda(x,t) := \left(\frac{1}{4}|\nabla\tau|^2_{g_{\varphi(t)}} + |\mathrm{Rm}(x,t)|^2_{g_{\varphi(t)}}\right)^{\frac{1}{2}}$ will imply bounds on all covariant derivatives of Rm and τ. Then, the flow will exist as long as $\Lambda(x,t)$ remains bounded.

$\rightsquigarrow$ uniqueness and compactness theory [Lotay-Wei].

- Non-collapsing under the assumption of bounded Scal [G. Chen].

Solutions to the LF

Study of explicit solutions on

• simply connected Lie groups with left-invariant closed G_2-structure [Fernández-F-Manero; Lauret; F-Raffero].

• $\mathbb{T}^7$ with cohomogeneity one closed G_2-structure [Huang-Wang-Yao].

• $M^6 \times S^1$, with a warped closed G_2-structure $\varphi = f\, ds \wedge \omega + \rho$, $f \in \mathcal{C}^\infty(M^6)$, $f > 0$ and compact base (M^6, ω, ρ) [F-Raffero].

• M^7 with an S^1-invariant closed G_2-structure [Fowdar].

• $(M^4 \times T^3, \varphi)$, where φ is induced by a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on the compact M^4 [Fine-Yao].

• Long-time existence and convergence of LF flow in cases related to coassociative fibrations [Lambert-Lotay].

Laplacian solitons

Self-similar solutions $\varphi(t) = \sigma(t) f_t^* \varphi$ of the LF $\Longleftrightarrow$ closed G_2-structures φ satisfying

$$\Delta_\varphi \varphi = \lambda \varphi + L_X \varphi \quad \text{(Laplacian solition)}$$

for some $\lambda \in \mathbb{R}$ and vector field X.

Definition

A Laplacian soliton φ is called

- shrinking if $\lambda < 0$,
- steady if $\lambda = 0$,
- expanding if $\lambda > 0$.

Theorem (Lin; Lotay-Wei)

On a compact manifold any Laplacian soliton φ (which is not torsion-free) must have $\lambda > 0$ and $X \neq 0$.

In particular, on a compact 7-manifold the only steady Laplacian solitons are given by parallel G_2-structures.

Open Problem

$\exists$ expanding Laplacian solitons on compact manifolds?

In the non-compact case:

- $\exists$ steady, shrinking and expanding (homogeneous) solitons [Lauret-Nicolini; F-Raffero; Ball].
- $\exists$ inhomogeneous complete steady and shrinking gradient solitons [Ball; Fowdar].

Semi-algebraic Laplacian solitons

Any homogeneous Laplacian soliton φ on a Lie group G is a semi-algebraic soliton, i.e. X is defined by a 1-parameter group of automorphisms induced by a derivation D of $\mathfrak{g}$ [Lauret].

Theorem (F, Raffero, Salvatore)

Let $\mathfrak{g}$ a unimodular Lie algebra with $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ admitting a semi-algebraic soliton φ. Then

- *if $\mathfrak{g}$ is the contactization of a symplectic Lie algebra, then $\lambda = |\tau|^2 \hookrightarrow \varphi$ is expanding;*
- *if $\dim \mathfrak{z}(\mathfrak{g}) = 2 \hookrightarrow$ 10 isomorphism classes (7 are nilpotent).*

Remark

The known examples of Lie groups admitting shrinking or steady Laplacian solitons have trivial center!

Exact G_2-structures

Remark

An expanding Laplacian soliton is an exact G_2-structure!

Problem

Does there exist compact $\Gamma\backslash G$ with an invariant exact G_2-structure?

A unimodular Lie group cannot admit any left-invariant exact symplectic form [Diatta-Manga].

Example (Fernández-F-Raffero)

There exists a unimodular solvable Lie algebra $\mathfrak{s} = \mathbb{R} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ 4-step nilpotent, satisfying $b_2(\mathfrak{s}) = b_3(\mathfrak{s}) = 0$ and admitting an exact G_2-structure.

Remark

The simply connected solvable Lie group S with Lie algebra $\mathfrak{s}$ is not strongly unimodular $\Rightarrow$ S does not admit any compact quotient $\Gamma\backslash S$!

Definition (Garland)

A solvable G is strongly unimodular if $\mathrm{tr}(\mathrm{ad}_X)|_{\mathfrak{n}^i/\mathfrak{n}^{i+1}} = 0$, for every $X \in \mathfrak{g}$, where $\mathfrak{n}^0 = \mathfrak{n}, \mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}], i \geq 1$, is the descending central series of the nilradical $\mathfrak{n}$ of $\mathfrak{g}$.

There are no compact examples $\Gamma\backslash G$ with an invariant exact G_2-structure, if $\mathfrak{g}$ is (2,3)-trivial, i.e. if $b_2(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$.

Theorem (Fernández-F-Raffero)

A strongly unimodular $(2,3)$-trivial $\mathfrak{g}$ does not admit any exact G_2-structure.

To prove the result:

- we use the property that a $(2,3)$-trivial $\mathfrak{g}$ is solvable and $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ nilpotent [Madsen-Swann]
- we classify 7-dim strongly unimodular $(2,3)$-trivial Lie algebras.

Problem

What happens if either $b_3(\mathfrak{g}) \neq 0$ or $b_2(\mathfrak{g}) \neq 0$?

Theorem (Freibert, Salamon)

If the Lie algebra of G has a codimension-one nilpotent ideal, then $\Gamma\backslash G$ does not admit any invariant exact G_2-structure.
If in addition G is completely solvable, $\Gamma\backslash G$ does not have any exact G_2-structure at all.

ERP Condition

Theorem (Cleyton-Ivanov; Bryant)

If M is compact with a closed G_2-structure φ, then

1) g_φ Einstein $\Rightarrow \tau \equiv 0$, i.e. φ is parallel.

2) $\int_M [Scal(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |Ric(g_\varphi)|^2 dV_\varphi$.

[Bryant]: equality in 2) holds if and only if

$$d\tau = \frac{|\tau|^2}{6}\varphi + \frac{1}{6} * (\tau \wedge \tau),$$

in such a case, φ is called extremally Ricci pinched (ERP).

Theorem (F-Raffero)

M compact with an ERP closed G_2-structure φ. Then the solution of the Laplacian flow with initial condition $\varphi(0) = \varphi$ is defined for every $t \in \mathbb{R}$ and remains ERP.

Example (Kath-Lauret)

A compact locally homogeneous space with an ERP G_2-structure is given by the compact quotient of the unimodular solvable Lie group S with structure equations

$$(0, 0, 0, -e^{14} - e^{24} - e^{34}, -e^{15} + e^{25} + e^{35}, e^{16} - e^{26} + e^{36}, e^{17} + e^{27} - e^{37})$$

by a lattice.

S is the only unimodular Lie group admitting a left-invariant ERP G_2-structure [F-Raffero].

THANK YOU VERY MUCH FOR THE ATTENTION!!