On a problem of M. Talagrand

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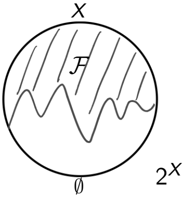
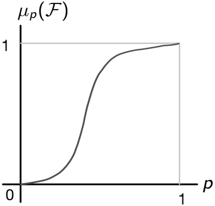
Definitions

- X: finite set; $2^X = \{\text{subsets of } X\}$; $[n] := \{1, 2, \ldots, n\}$
- μ_p: p-biased product probability measure on 2^X

$$\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|} \quad A \subseteq X$$

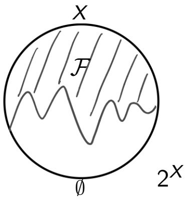
- $X_p \sim \mu_p$ (e.g. $X = \binom{[n]}{2} \to X_p = G_{n,p}$)
- $\mathcal{F} \subseteq 2^X$ is an increasing property if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$
  - graphic e.g.'s: $\mathcal{F} = \{\text{connected}\}$; $\mathcal{F} = \{\text{contain a triangle}\}$
- Assume $\mathcal{F} \neq \emptyset, 2^X$.
- Fact. Given $\mathcal{F}$, $\mu_p(\mathcal{F})(= \mathbb{P}(X_p \in \mathcal{F}))$ is strictly increasing in p

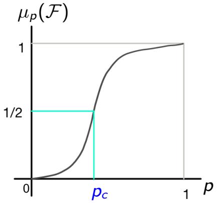




Definitions

- The threshold $p_c(\mathcal{F})$: $\mu_{p_c}(\mathcal{F}) = 1/2$ (unique)





Central question in probabilistic combinatorics :

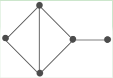
Given $\mathcal{F}$, what's the threshold for $\mathcal{F}$?

E.g. $G_{n,p}$: threshold for connectivity, Hamiltonicity, etc.

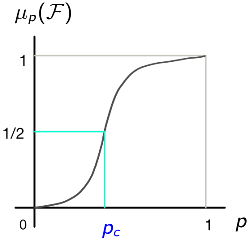
Getting a lower bound – expectation calculation

Example 1.

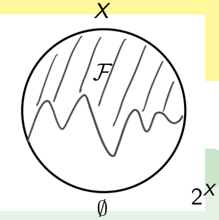
$X = \binom{[n]}{2}$ ($X_p = G_{n,p}$), $\mathcal{F}$: contain K



Q. $p_c(\mathcal{F})$ (the threshold for $G_{n,p}$ to contain K)?



$q(\mathcal{F})$: the expectation threshold



- Given $\mathcal{G} \subseteq 2^X$, $\langle \mathcal{G} \rangle := \{T : \exists\, S \in \mathcal{G}, S \subseteq T\}$

Observation

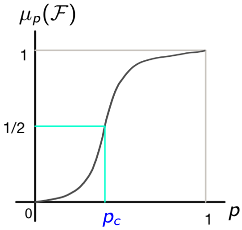
We have $p \leq p_c(\mathcal{F})$ if $\exists\, \mathcal{G} \subseteq 2^X$ such that

(1) $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$ ("$\mathcal{G}$ covers $\mathcal{F}$")

(2) $\sum_{S \in \mathcal{G}} p^{|S|} \leq \frac{1}{2}$

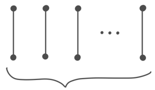
$(\because)$

- Ex 1. $\mathcal{F} = \{\text{contain } K\}$
  - $\mathbb{E}(\#\text{K's}) \approx n^5 p^6 \to 0$ if $p \ll n^{-5/6}$
  - $\mathbb{E}(\#\text{H's}) \approx n^4 p^5 \to 0$ if $p \ll n^{-4/5}$



- the expectation threshold $q(\mathcal{F}) := \max\{p : \exists\, \mathcal{G}\}$ $\to q(\mathcal{F}) \leq p_c(\mathcal{F})$

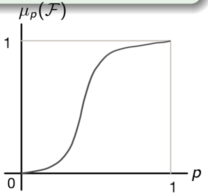
Another lower bound – Coupon collector



Example 2.

$X = \binom{[n]}{2}$ $(X_p = G_{n,p})$, $\mathcal{F}$: contain a perfect matching

Q. $p_c(\mathcal{F})$ (the threshold for $G_{n,p}$ to contain a PM)?



- the expectation threshold $q(\mathcal{F}) \approx 1/n$
- BUT: $p_c(\mathcal{F}) \approx \log n/n$
- Actually, $\log n/n$ is another lower bound on $p_c(\mathcal{F})$.

$$p \ll \log n/n \Rightarrow G_{n,p} \text{ has an isolated vertex w.h.p.}$$

Kahn-Kalai Conjecture ('07)

$\exists$ a universal $K > 0$ such that for any finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq Kq(\mathcal{F}) \log |X|.$$

$q_f(\mathcal{F})$: the fractional exp thr

Observation

We have $p \le p_c(\mathcal{F})$ if $\exists\, \mathcal{G} \subseteq 2^X$ such that

(1) $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$ ("$\mathcal{G}$ covers $\mathcal{F}$")

(2) $\sum_{S \in \mathcal{G}} p^{|S|} \le \frac{1}{2}$

More observation

We have $p \le p_c(\mathcal{F})$ if $\exists\, \lambda : 2^X \to [0, \infty)$ such that

1. $\sum_{S \subseteq F} \lambda(S) \ge 1 \quad \forall\, F \in \mathcal{F}$
2. $\sum_{S \subseteq X} \lambda(S) p^{|S|} \le \frac{1}{2}$

- the fractional expectation threshold $q_f(\mathcal{F}) := \max\{p : \exists\, \lambda\}$
- Easy. $q(\mathcal{F}) \le q_f(\mathcal{F}) \le p_c(\mathcal{F})$

Theorem. (Frankston-Kahn-Narayanan-P. '19)

$\exists$ a universal $K > 0$ such that for any finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \le K q_f(\mathcal{F}) \log |X|.$$

Weaker than Kahn-Kalai Conjecture (but still very strong)

Talagrand's conjecture on $q(\mathcal{F})$ vs. $q_f(\mathcal{F})$

Conjecture (Talagrand '10)

There exists a universal constant $L > 0$ such that for any finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$q_f(\mathcal{F}) \leq Lq(\mathcal{F})$$

- Implies KK Conj (via FKNP Thm)
- Even simple instances of the conjecture are not easy to establish.
- True if λ is supported on singletons (Talagrand '10)
- Talagrand suggested two test cases:
  - $X = \binom{[n]}{2} = E(K_n)$ and (for some k) λ is the indicator of copies of K_k in K_n (DeMarco-Kahn '15)
  - λ is supported on 2-element sets (Frankston-Kahn-P. '21)

Conjecture (Talagrand '10)

$\exists L > 0$ such that for any finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$q_f(\mathcal{F}) \leq Lq(\mathcal{F})$$

Two possible approaches: either give

1. Upper bound on $q_f(\mathcal{F})$; or
2. Lower bound on $q(\mathcal{F})$

Reformulation

$\exists L > 0$ such that for any finite X, $p \in [0, 1]$, and $\lambda : 2^X \to [0, \infty)$,

$$\mathcal{F} := \{U \subseteq X : \sum_{S \subseteq U} \lambda_S \geq \sum_{S \subseteq X} \lambda_S (Lp)^{|S|}\}$$

admits $\mathcal{G} \subseteq 2^X$ that satisfies

1. $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$
2. $\sum_{S \in \mathcal{G}} p^{|S|} \leq \frac{1}{2}$

Open question

Conjecture (Talagrand, '10)

$\exists L > 0$ such that for any finite X, $p \in [0,1]$, and $\lambda : 2^X \to [0, \infty)$,

$$\mathcal{F} := \{U \subseteq X : \sum_{S \subseteq U} \lambda_S \geq \sum_{S \subseteq X} \lambda_S (Lp)^{|S|}\}$$

admits $\mathcal{G} \subseteq 2^X$ that satisfies

1. $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$
2. $\sum_{S \in \mathcal{G}} p^{|S|} \leq \frac{1}{2}$

Thank you!