Flows in infinite networks

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Mathematical models for interacting dynamics on networks

Joint work

Christian Budde, M.K.F, *Bi-continuous semigroups for flows in infinite networks*, J. Networks Heterogeneous Media, to appear.

Infinite networks, metric graphs

Infinite networks, metric graphs

$G = (V, E)$, simple, locally finite

$\mathrm{e}_j \simeq [0, 1]$

$$\mathbb{B}_{ij} = \begin{cases} w_{ij}, & e_j(0) = e_i(1) \\ 0, & \text{else} \end{cases}$$

Flows in networks

Flows in networks

- on every edge e_j:

$$\frac{d}{dt}u_j(x,t) = \frac{d}{dx}u_j(x,t) \tag{TE}$$

- in every vertex v_i:

$$u_j(1,t) = \sum_{k \in J} \mathbb{B}_{jk} u_k(0,t) \tag{BC}$$

- at $t = 0$:

$$u_j(x,0) = f_j(x) \tag{IC}$$

$$(TE) + (BC) + (IC) = (F)$$

Flows in networks

1996 Barletti, 2005 $\rightsquigarrow$ KF & Sikolya, Matrai, Radl, Dorn, Keicher, Banasiak, Puchalska, Namayanja, ...

Semigroup approach

- $X := \mathrm{L}^1([0,1],\mathbb{C}^m)$ or $X := \mathrm{L}^1([0,1],\ell^1)$
- $A := \mathrm{diag}(\frac{d}{dx})$, $D(A) := \{f \in \mathrm{W}^{1,1} \mid f(1) = \mathbb{B}f(0)\}$
- $(F) \iff (ACP): \dot{u} = Au, u(0) = u_0$
- A generates strongly continuous semigroup $(T(t))$ on X:

$$T(t)f(x) = \mathbb{B}^n f(t+x-n), n \leq t+x < n+1, n \in \mathbb{N}_0 \quad (1)$$

Bi-continuous semigroups

Bi-continuous semigroups

2001 $\rightsquigarrow$ Kühnemund, Farkas, Albanese, Lorenzi, Budde, ...

Assumptions

X Banach space with norm $\|\cdot\|$ & locally convex topology τ s.t.

(i) τ is Hausdorff, coarser then $\|\cdot\|$-topology

(ii) every $\|\cdot\|$-bounded τ-Cauchy sequence in τ-convergent

(iii) $\|f\| = \sup_{\varphi \in (X,\tau)', \|\varphi\| \leq 1} |\varphi(f)|$

Bi-continuous semigroups

Definition

$(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a *bi-continuous semigroup* on X if

(i) $T(t+s) = T(t)T(s)$ and $T(0) = I$, $s, t \geq 0$

(ii) $t \mapsto T(t)f$ τ-continuous for every $f \in X$

(iii) $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$

(iv) if $\|f_n\| < \infty$, $f_n \xrightarrow{\tau} 0$ then $T(s)f_n \xrightarrow{\tau} 0$-uniformly for $s \in [0, t_0]$

Its *generator:*

$$
\begin{aligned}
Af &:= \tau - \lim_{t\to 0} \frac{T(t)f - f}{t} \\
D(A) &:= \left\{ f \mid \tau - \lim \text{ exists and } \sup_{t\in(0,1]} \frac{\|T(t)f - f\|}{t} < \infty \right\}
\end{aligned}
$$

L^∞-wellposedness of (F)

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Theorem

The operator

$$A := \mathrm{diag}\left(\frac{d}{dx}\right), \quad D(A) := \{f \in \mathrm{W}^{1,\infty} \mid f(1) = \mathbb{B}f(0)\},$$

generates a contraction bi-continuous semigroup on $\mathrm{L}^\infty([0,1],\ell^1)$ *with respect to the weak***-topology. This semigroup is given in* (1).

Generalisations and further results

- ✓ include velocities in (TE): $c_j \frac{d}{dx} u_j(x,t)$ or $c_j(x) \frac{d}{dx} u_j(x,t)$
- ✓ include absorption term in (TE): $q_j(x) u_j(t,x)$
- ✓ consider general matrix $\mathbb{B}$
- ✓ study long-time behaviour (Dobrick, 2021)
- × study further properties (stability, control,...)