First order Mean Field Games on networks

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Outline

- A brief introduction to Mean Field Games
- Definition of networks
- A MFG problem on networks with control on the velocity
- Work in progress: control on the acceleration (with/without constraint on the control)

A brief introduction to Mean Field Games

The Mean Field Games (MFG) theory was proposed by Lasry-Lions, and independently by Huang-Malhamé-Caines, in 2006 for modelization of interactions among a very large ("infinite") number of agents when individual actions are related to mass behaviour and vice versa.

Applications: financial markets, fashion trends, pedestrian or vehicular traffic...

Distinctive features of the model:

- The agents are influenced only by the average behaviour of all other players (in analogy with Statistical Mechanics).
- The agents are rational: they choose a strategy so to minimize a cost.
- The agents are indistinguishable.
- The agents are individually neglectable: a single agent by itself cannot influence the collective behaviour.

Model example

Consider a game with N players. The i-th player's dynamics is

$$dX_s^i = \alpha_s^i ds + \sqrt{2\nu}\, dW_s^i, \qquad X_t^i = x \in \mathbb{R}^n$$

where $\nu \geq 0$, W^i are independent Brownian motions, while α^i is the control chosen so to minimize the cost functional

$$\mathbb{E}\left\{ \int_t^T \left[\frac{|\alpha_s^i|^2}{2} - \ell(X_s^i, s) + F\Big[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}\Big](X_s^i) \right] ds \right.$$
$$\left. + G\Big[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_T^j}\Big](X_T^i) \right\}.$$

The Nash equilibria are characterized by a system of $2N$ equations. Nevertheless, as $N \to +\infty$, this system reduces to the following one:

MFG system

$$
\begin{cases}
-\partial_t u - \nu \Delta u + \frac{1}{2}|\nabla u|^2 + \ell(t,x) = F[m(t)](x) & (t,x) \in (0,T) \times \mathbb{R}^n \\
\partial_t m - \nu \Delta m + \operatorname{div}(m \nabla u) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\
u(T,x) = G[m(T)](x) & x \in \mathbb{R}^n \\
m(0,x) = m_0(x) & x \in \mathbb{R}^n
\end{cases}
$$

where m_0 is the initial distribution of players: $m_0 \geq 0$, $\int_{\mathbb{R}^n} m_0 dx = 1$.

- The first equation is a backward-in-time Hamilton-Jacobi(-Bellman) equation describing the expected value for a generic player.
- The second equation is a forward-in-time Fokker-Planck/continuity equation describing the density m of the players.
- Three couplings occur between the equations.

Variants / other applications

- the costs F and G may depend on m in a local/nonlocal way;
- infinite horizon problem;
- dominant single player versus a population of small players;
- several populations of identical agents;
- cost depending on the velocity of other players and not on their positions;
- penalization of mass concentration;
- all players follow the same feedback law (*Mean Field Type Control*);
- the generic agent controls its acceleration (and $\nu = 0$);
- the agents' positions are constrained in a closed subset of $\mathbb{R}^n$.

First order case, i.e. $\nu = 0$

$$
\begin{cases}
(HJ) & -\partial_t u + \frac{1}{2}|\nabla u|^2 + \ell(t,x) = F[m(t)](x) & (t,x) \in (0,T) \times \mathbb{R}^n \\
(C) & \partial_t m + \mathrm{div}\,(m \nabla u) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\
& u(T,x) = G[m(T)](x) & x \in \mathbb{R}^n \\
& m(0,x) = m_0(x) & x \in \mathbb{R}^n
\end{cases}
$$

Definition

$(u,m) \in W^{1,\infty}_{\mathrm{loc}}([0,T] \times \mathbb{R}^n) \times C([0,T]; \mathcal{P}_1(\mathbb{R}^n))$ is a solution if:

-) (HJ)-equation is satisfied by u in the viscosity sense

-) (C)-equation is satisfied by m in the sense of distributions.

Theorem (Cardaliaguet - PL Lions)

1. *The MFG system has a solution* (u, m)*;*
2. $m(x,s) = \Phi(x,0,s)\# m_0(x)$*, where* Φ *is the flow of the dynamics*

$$(1) \qquad x'(s) = -\nabla u(x(s),s), \qquad x(0) = x.$$

Ingredients of the proof

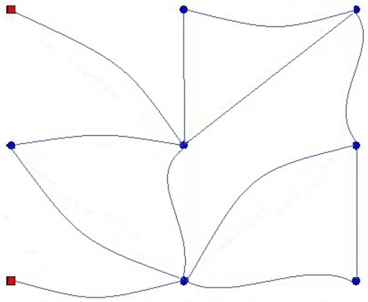
i) for a.e. x, optimal trajectories may bifurcate only at initial time;
ii) the optimal controls are bounded uniformly w.r.t. x;
iii) the value function is Lipschitz continuous and semiconcave;
iv) for a.e. x, system (1) describes the (unique) optimal trajectory of the optimal control problem;
v) Schauder fixed point theorem.

Literature for 1^{st} order MFG

- MFG on Euclidean spaces
  - *classical approach*
    - P.L. Lions' lectures at Collége de France 2012 - Cardaliaguet "Notes on Mean Field Games",
    - Cardaliaguet, DGA 2013
    - Gomes-Pimentel-Voskanyan, SpringerBrief 2016
  - *Lagrangian approach*
    - Benamou-Carlier-Santambrogio, Springer 2016
    - Cannarsa-Capuani, Springer-Indam 28, 2018
    - Mazanti-Santambrogio, M^3AS 2019
- MFG on discrete sets
  - Gomes-Mohr-Souza, JMPA 2010
  - Gomes-Mohr-Souza, AMO 2013
  - Guéant, AMO 2015
- MFG on networks (all for 2^{nd} order case)
  - Camilli-M. , SIAM JCO 2016
  - Achdou-Dao-Ley-Tchou, NHM 2019
  - Achdou-Dao-Ley-Tchou, CVPDE 2020

Network

A network is a connected, embedded in $\mathbb{R}^n$, set $\mathcal{N}$ and it is formed by a set of vertices $V := \{v_i\}_{i \in I}$ and a set of regular edges $E := \{e_j\}_{j \in J}$. We assume that the network is compact and without boundary.



(a) An example of network

Notations

- $Inc_i := \{j \in J : \ e_j \text{ incident to } v_i \in V\}$.
- Any edge e_j is parametrized by a smooth function $\pi_j : [0, l_j] \to \mathbb{R}^n$. For a function $u : \mathcal{N} \to \mathbb{R}$ we denote by $u_j : [0, l_j] \to \mathbb{R}$ its restriction to e_j, i.e. $u(x) = u_j(y)$ for $x \in e_j$, $y = \pi_j^{-1}(x)$.
- The derivative is considered w.r.t. the parametrization.
- In $v_i \in V$, the oriented derivative of u is

$$\partial_j u(v_i) := \begin{cases} \lim_{h\to 0^+}[u_j(h) - u_j(0)]/h, & \text{if } v_i = \pi_j(0) \\ \lim_{h\to 0^+}[u_j(l_j - h) - u_j(l_j)]/h, & \text{if } v_i = \pi_j(l_j) \end{cases}$$

and $D_x u(v_i) := (\partial_j u(v_i))_{j\in Inc_i}$.

MFG on networks

Dynamics of a generic player

The state of a generic player is constrained in the network and, when it is inside an edge e_j, it obeys to

$$x'(t) = \alpha(t)$$

where α is the control.

Cost for the generic player

The generic player aims at choosing $\alpha \in L^2$ so to minimize the cost

$$J(x,t,\alpha) = \int_t^T \left[\frac{|\alpha(s)|^2}{2} - \ell(x(s),s) + F[m(s)](x(s))\right] ds + G[m(T)](x(T))$$

where $m(s)$ is the distribution of the whole population at time s.

Notations

- $\Gamma = AC(0,T;\mathcal{N})$
- $\Gamma[x] = \{\gamma \in \Gamma : \ \gamma(0) = x\}$
- $\mathcal{P}(\Gamma) = \{\text{Borel probability measures on } \Gamma\}$
- $\forall t \in [0,T]$, the evaluation map is $e_t : \Gamma \to \mathcal{N}$ with $e_t(\gamma) = \gamma(t)$
- $\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{P}(\Gamma) : \ e_0\#\eta = m_0\}$
- for each $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we set

$$\begin{aligned} J^\eta(t,x,\alpha) \ &= \ \int_t^T \left[\frac{|\alpha(s)|^2}{2} - \ell(\gamma(s),s) + F[e_s\#\eta](\gamma(s))\right] ds \\ &\quad + G[e_T\#\eta](\gamma(T)) \end{aligned}$$

where $\gamma(t) = x$ and $\gamma' = \alpha$ and

$$\Gamma^\eta[x] \ = \ \left\{\gamma \in \Gamma[x] : \ J^\eta(0,x,\gamma') \le J^\eta(0,x,\tilde{\gamma}') \quad \forall \tilde{\gamma} \in \Gamma[x]\right\}.$$

Definition

A measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a MFG equilibrium for m_0 if

$$\text{supp}(\eta) \subset \bigcup_{x \in \text{supp}(m_0)} \Gamma^{\eta}[x].$$

Theorem

Assume

- $m_0 \in \mathcal{P}(\mathcal{N})$
- $\ell \in C^0(\mathcal{N})$
- $F[\cdot], G[\cdot] : \mathcal{P}(\mathcal{N}) \to C^0(\mathcal{N})$ are bounded and continuous.

Then, there exists a MFG equilibrium η for m_0.

Proof (sketch)
Following the Lagrangian approach of [Cannarsa-Capuani, '18], we introduce the multivalued map

$$E : \mathcal{P}_{m_0}(\Gamma) \to \mathcal{P}_{m_0}(\Gamma)$$

$$E(\eta) = \left\{ \hat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : \quad \mathrm{supp}(\hat{\eta}) \subset \bigcup_{x \in \mathrm{supp}(m_0)} \Gamma^{\eta}[x] \right\}$$

and we apply Kakutani fixed point Theorem to obtain a MFG equilibrium. Indeed, there holds

a) $\forall \eta \in \mathcal{P}_{m_0}(\Gamma)$, $E(\eta)$ is a nonempty set
b) $\forall \eta \in \mathcal{P}_{m_0}(\Gamma)$, $E(\eta)$ is a convex set
c) the map E fulfills the closed graph property.

Definition

A couple (u, m) is a mild solution to the MFG if there exists a MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that

- $m(t) = e_t \# \eta \qquad \forall t \in [0, T]$
- u is the the value function associated to η:

$$u(t, x) = \inf_{\alpha \text{ adm.}} J^\eta(t, x, \alpha).$$

Corollary

There exists a mild solution (u, m) to the MFG.

Hamilton-Jacobi problem for u

$$\begin{cases} -\partial_t u + \frac{1}{2}|\partial_j u|^2 + \ell = F[m(t)] & (t,x) \in (0,T) \times e_j \\ -\partial_t u + \max_{j \in Inc_i} \{\frac{1}{2}[(\partial_j u)_-]^2\} + \ell = F[m(t)] & (t, v_i) \in (0,T) \times V \\ u(T,x) = G[m(T)](x) & x \in \mathcal{N}. \end{cases}$$

Definition (viscosity solution)

u is a subsolution (resp., a supersolution) if: for all $\varphi \in C^1((0,T) \times \mathcal{N})$ s.t. $u - \varphi$ has a maximum (resp., a minimum) at (t,x), there holds

$$-\partial_t \varphi(t,x) + \frac{|\partial_j \varphi(t,x)|^2}{2} + \ell(t,x) \leq (\geq) F[m(t)](x) \qquad \text{if } x \in e_j$$

$$-\partial_t \varphi(t,x) + \max_{j \in Inc_i} \{\frac{[(\partial_j \varphi(t,x))_-]^2}{2}\} + \ell(t,x) \leq (\geq) F[m(t)](x) \qquad \text{if } x \in V.$$

u is a solution when it is both a sub- and a supersolution.

Proposition

u is the viscosity solution to the Hamilton-Jacobi problem.

Work in progress: control on the acceleration

Dynamics of a generic player

Inside an edge e_j, the state of a player obeys to

$$x'(t) = v(t), \qquad v'(t) = \alpha(t)$$

where the control α is chosen either. Two cases:

- α is chosen in $\mathbb{R}$
- α is chosen in $[-1, 1]$.

Cost for the generic player

$$J(x, v, t, \alpha) = \int_t^T \left[\frac{|\alpha(s)|^2}{2} - \ell(x(s), v(s), s) + F[m(s)](x(s), v(s)) \right] ds + G[m(T)](x(T), v(T)).$$

Difficulties. Inertia of dynamics, viability set, ...

Thank You!