Quantum isomorphism of graphs: an overview

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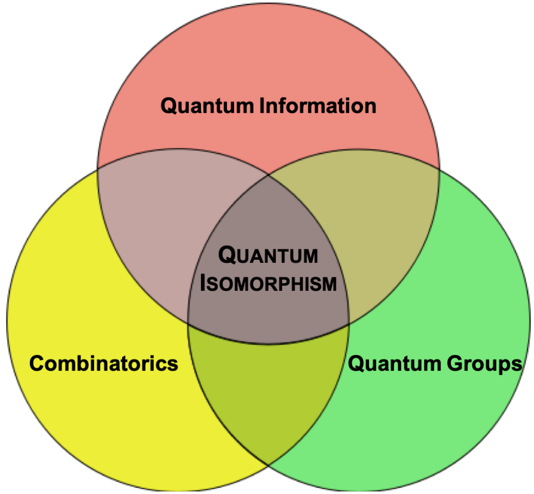
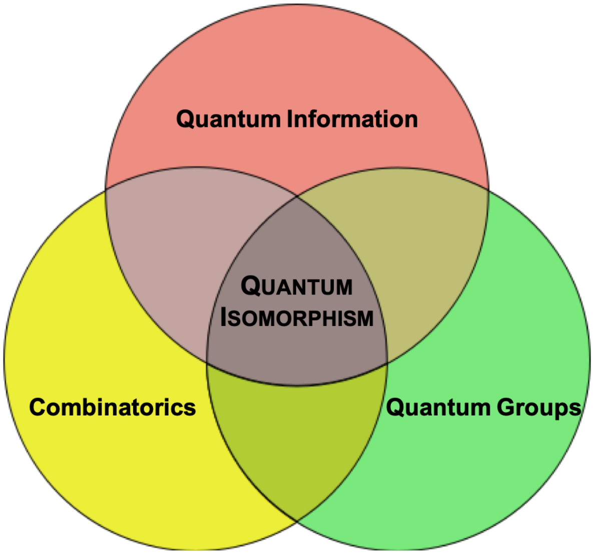


Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs.
Mančinska, Roberson.
Proceedings of FOCS'20.

Nonlocal games and quantum permutation groups.
Lupini, Mančinska, and Roberson.
Journal of Functional Analysis, **279(5)**:108592, 2020.

Quantum and non-signalling graph isomorphisms.
Atserias, Mančinska, Roberson, Šámal, Severini, and Varvitisiotis.
Journal of Combinatorial Theory, Series B, **136**:289–328, 2019.
Proceedings of ICALP'17 , LIPIcs **80**, 76:1–76:14, 2017.



- **Nonlocal games** provide a general framework for studying entanglement
- **Problem:** Entanglement-assisted strategies for arbitrary nonlocal games are **hard to analyze**
- **Line of attack:** Focus on a **well-behaved** class of games

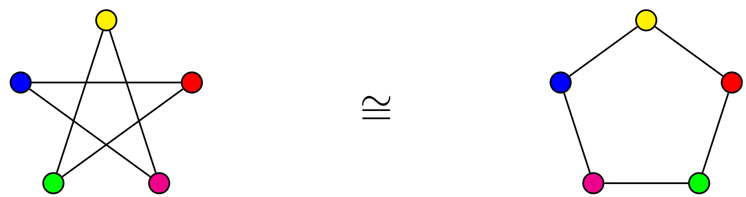
Overview

Quantum isomorphism = **operationally defined noncommutative variant of graph isomorphism**

We will see different yet equivalent ways to think about quantum isomorphism of graphs

- Nonlocal games
- Matrix formulations
- Homomorphism counts

Graph isomorphism



A map $f : V(G) \to V(H)$ is an **isomorphism** from G to H if

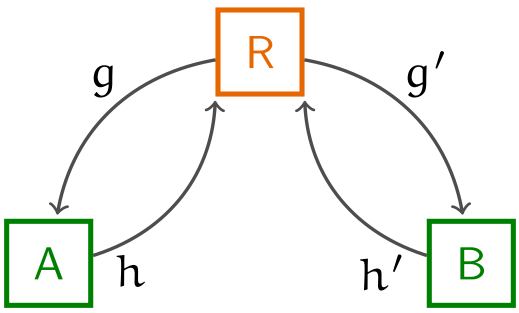
- f is a bijection and
- $g \sim g'$ if and only if $f(g) \sim f(g')$.

If such a map exists, we say that G and H are **isomorphic** and write $G \cong H$.

Matrix formulation: $PA_GP^\dagger = A_H$ for some **permutation** matrix P

(G, H)-Isomorphism Game

Intuition: Alice and Bob want to convince a referee that $G \cong H$.



- To win players must reply h, h' such that $\mathrm{rel}(h, h') = \mathrm{rel}(g, g')$
- No communication during game

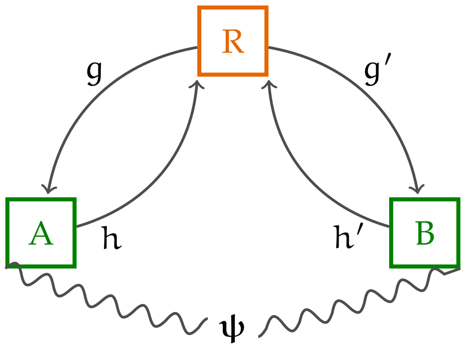
Fact. $G \cong H \Leftrightarrow$ **Classical** players can win the game with certainty

Def. (Quantum isomorphism)
We say that $G \cong_{qc} H$ if **quantum**[1] players can win the game with certainty.

[1] We work in the **commuting** rather than the tensor-product model.

Quantum commuting strategies

$G \cong_{qc} H :=$ **Quantum** players can win the (G, H)-isomorphism game

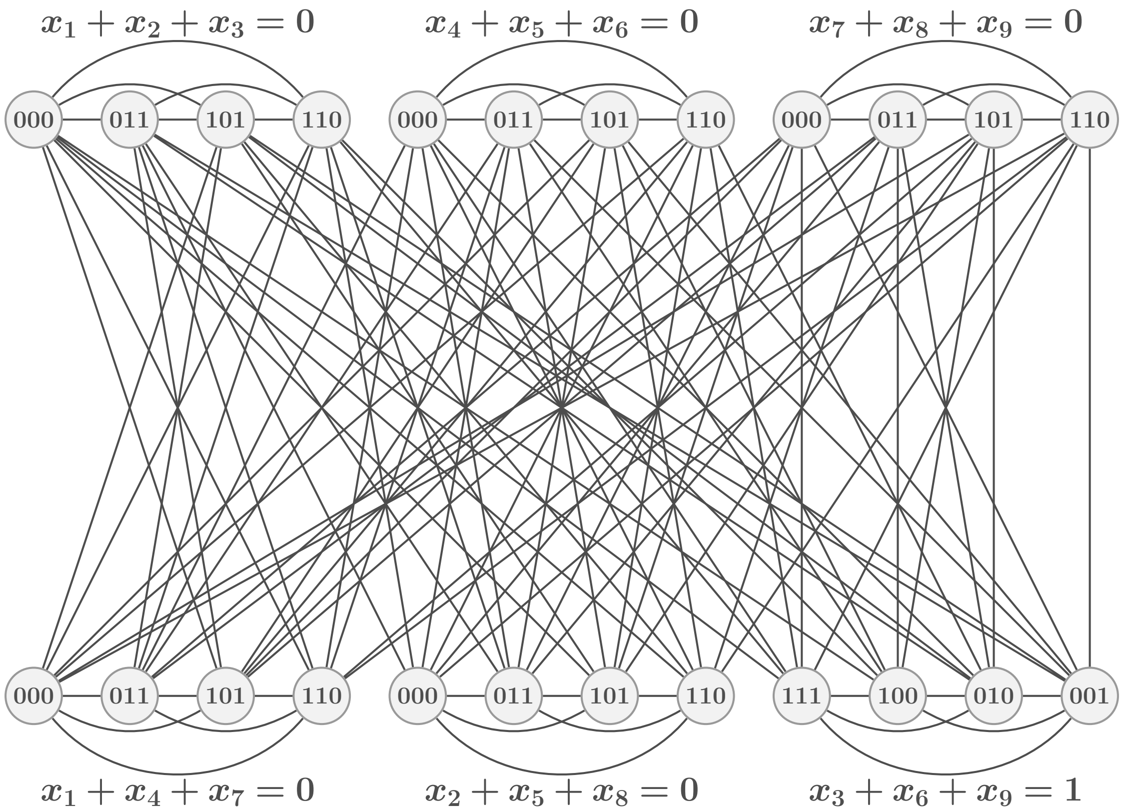


- Alice and Bob share a quantum state ψ
  ψ is a unit vector in a Hilbert space $\mathcal{H}$
- Upon receiving g, Alice performs a local measurement $\mathcal{E}_g$ to get $h \in V(H)$
  $\mathcal{E}_g = \{E_{gh} \in \mathcal{B}(\mathcal{H}) : h \in V(H)\}$ where
  $$E_{gh} \succeq 0, \quad \sum_h E_{gh} = I.$$
- Bob measures with $\mathcal{F}_{g'}$
- E_{gh} and $F_{g'h'}$ commute

The probability that players respond with h, h' on questions g, g' is

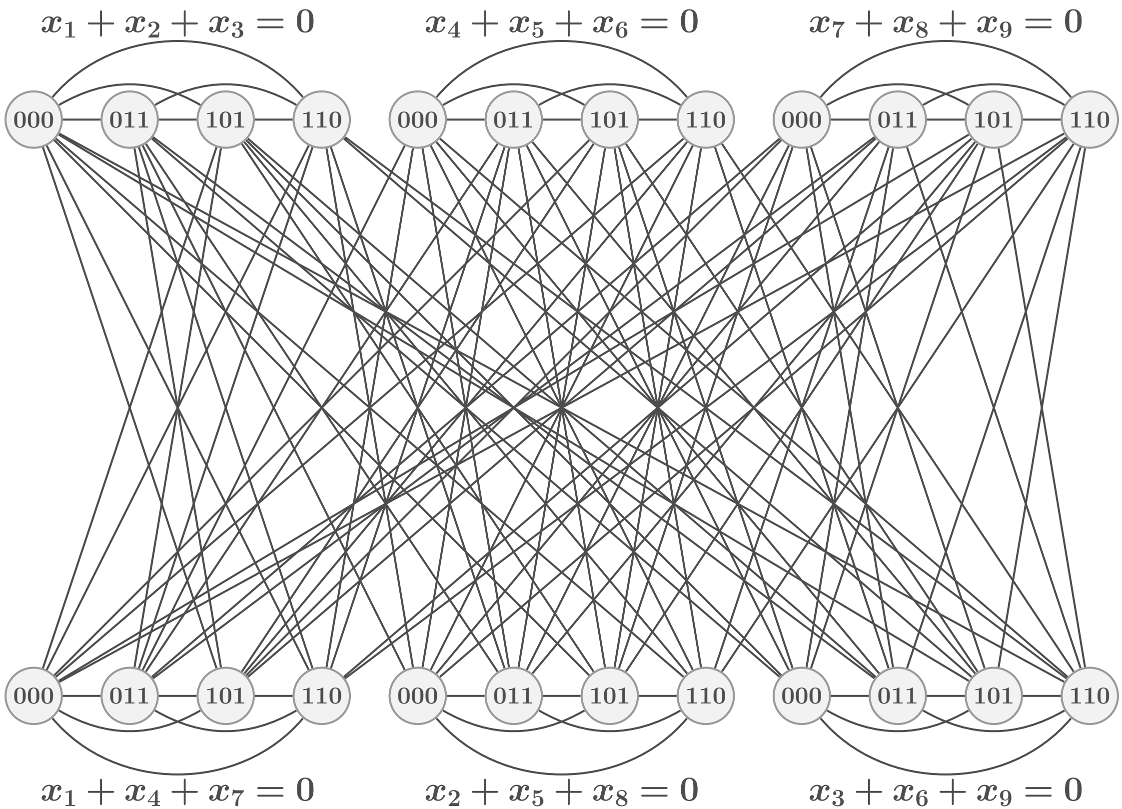
$$p(h, h'|g, g') = \langle \psi, \left(E_{gh} F_{g'h'}\right) \psi \rangle$$

Example: G $\not\cong$ H but G $\cong_{qc}$ H



Construction based on reduction from linear system games.

Example: G $\not\cong$ H but G $\cong_{qc}$ H



Construction based on reduction from linear system games.

Quantum isomorphism and quantum groups

Def. A matrix $\mathcal{P} = (p_{ij})$ whose entries are elements of a C^*-algebra is a **quantum permutation matrix** (QPM), if

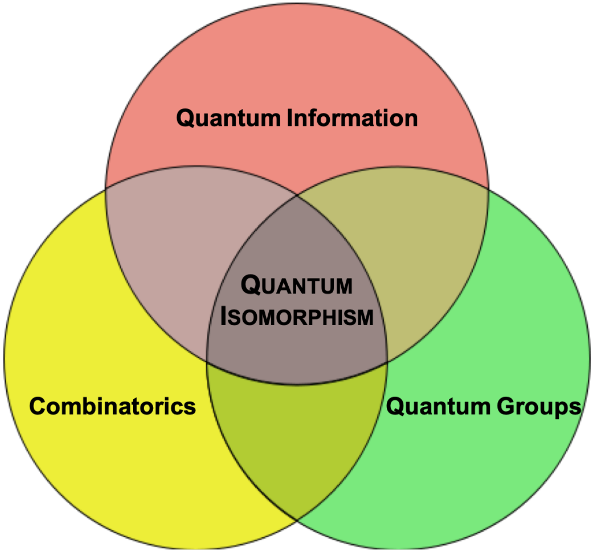
- p_{ij} is a projection, i.e., $p_{ij}^2 = p_{ij} = p_{ij}^*$ for all i, j
- $\sum_k p_{ik} = \mathbf{1} = \sum_\ell p_{\ell j}$ for all i, j

Remark. A QPM with entries from $\mathbb{C}$ is a permutation matrix.

Thm. (Lupini, M., Roberson)

$$G \cong_{qc} H \quad \Leftrightarrow \quad \mathcal{P} A_G \mathcal{P}^\dagger = A_H \text{ for some } \textbf{quantum permutation matrix } \mathcal{P}$$

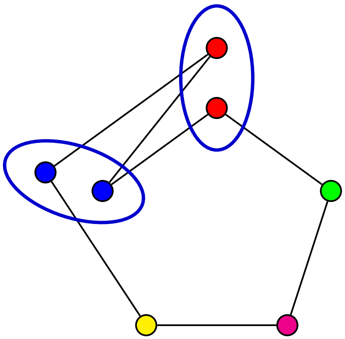
Can we describe quantum isomorphism in combinatorial terms?



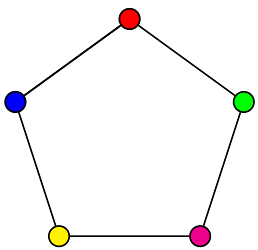
Graph homomorphisms

Def. A map $\varphi : V(F) \to V(G)$ is a **homomorphism** from F to G if $\varphi(u) \sim \varphi(v)$ whenever $u \sim v$.

Example



$C_7 \to C_5$



$\mathbf{hom}(F, G)$:= # of homomorphisms from F to G.

Counting homomorphisms

Theorem. (Lovász, 1967)
Homomorphism counts determine a graph up to isomorphism, i.e.

$$G \cong H \iff \hom(F, G) = \hom(F, H) \text{ for all graphs } F.$$

Theorem. (M., Roberson)
$G \cong_{qc} H \iff \hom(F, G) = \hom(F, H)$ for all **planar** graphs F.

Context: Homomorphism counting

Thm. (Lovász, 1967)
$G \cong H \;\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for **all graphs** F

Thm. (M., Roberson, 2019)
$G \cong_{qc} H \;\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for all **planar** graphs F

Folklore.
G and H cospectral $\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for all **cycles** F

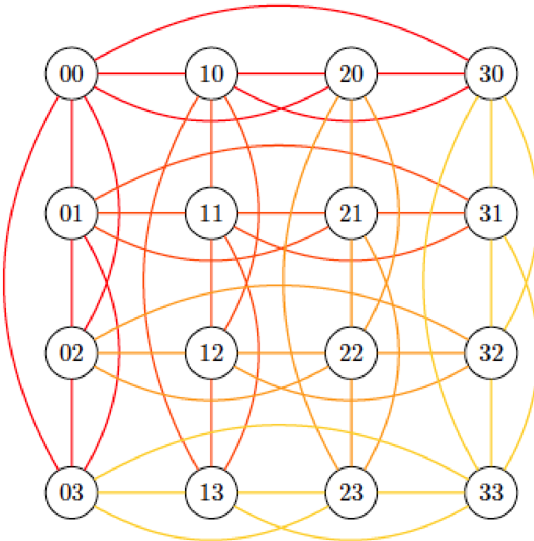
Thm. (Dvořák, 2010; Dell, Grohe, Rattan, 2018)
$G \cong_{f} H \;\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for all **trees** F
$G \cong_{k} H \;\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for all F of **treewidth** $\leqslant k$

Complexity: Except for the class of planar graphs, equality of homomorphism counts from all of the above graph classes can be tested in at worst quasi-polynomial time.

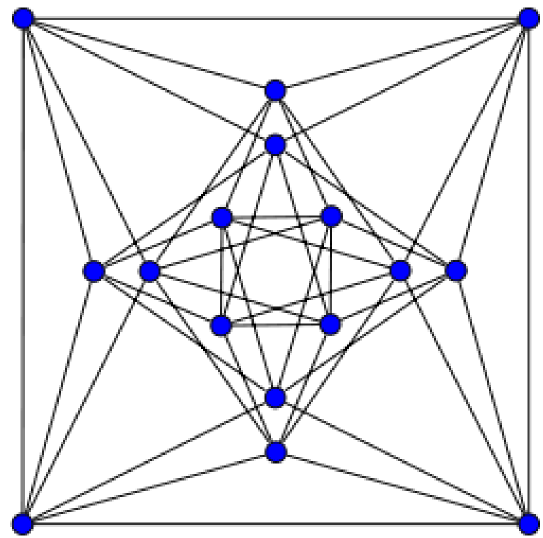
Application: Certificate for $G \not\cong_{qc} H$

Are these two graphs quantum isomorphic?

Rook graph



Shrikhande graph

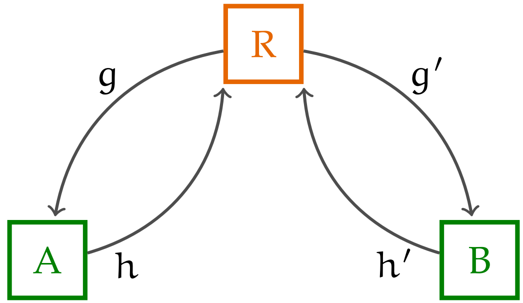


Before: Difficult to prove that they are not quantum isomorphic.

Now: Only one (the Rook graph) contains K_4.

Summary

Graph isomorphism can be formulated in terms of a **nonlocal game**.



- $G \cong_{qc} H :=$ **Quantum** players can win the isomorphism game
- **Thm.** $G \cong_{qc} H \quad \Leftrightarrow \quad \mathcal{P} A_G \mathcal{P}^\dagger = A_H$ for some **quantum permutation matrix** $\mathcal{P}$
- **Thm.** $G \cong_{qc} H \quad \Leftrightarrow \quad \hom(F, G) = \hom(F, H)$ for all **planar** F

Thank you!