Non-holonomic equations for sub-Riemannian extremals and metrizable parabolic geometries

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Sheaf $\mathcal{D}^{-1} = \mathcal{D}$ of vector fields valued in D generates the filtration by sheafs

$$\mathcal{D}^j = \{[X, Y], X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \dots .$$

We say that D is a bracket generating distribution if for some k, $\mathcal{D}^k$ is the sheaf of all vector fields on M.

Bracket generating distribution D defines the filtration of subspaces

$$T_xM = D_x^k \supset \cdots \supset D_x^{-1}$$

at each point $x \in M$.
The associated graded tangent space

$$\text{gr } T_xM = T_xM/D_x^{k+1} \oplus \cdots \oplus D_x^{-1}$$

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Definition

(M, D, S) is a *sub-Riemannian geometry with constant symbol* if D is bracket generating, and the nilpotent algebra $\operatorname{gr} T_x M$, **together with the metric**, is isomorphic to a fixed graded Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$$

with a fixed metric σ on $\mathfrak{g}_{-1}$.

The sub-Riemannian metric can be viewed as $h : T^*M \to TM$ with the image D and kernel $D^\perp$.
There are the equivalent short exact sequences:

$$0 \to K \to T^*M \xrightarrow{h} D \to 0$$

$$0 \to D \to TM \xrightarrow{q} Q \to 0.$$

There is also the D-valued Levi-form defined by projecting the Lie bracket of vector fields in D

$$L : D \times D \to Q.$$

Splittings of the sequences correspond to splittings of TM or T^*M.

Tractor-like view

A change of splitting from s to another $\widehat{s} : Q \to TM$ may be naturally identified with a bundle map $f : Q \to D$.
Changes of splitting induce:

$$[TM]_s \ni [v]_s = \begin{pmatrix} \sigma^a \\ u^i \end{pmatrix}_s \mapsto \begin{pmatrix} \widehat{\sigma}^a \\ \widehat{u}^i \end{pmatrix}_{\widehat{s}} = \begin{pmatrix} \sigma^a \\ u^i - f_a^i \sigma^a \end{pmatrix}_{\widehat{s}} = [v]_{\widehat{s}} \in [TM]_{\widehat{s}}$$

and similarly

$$\begin{pmatrix} u^i \\ \nu_a \end{pmatrix} \mapsto \begin{pmatrix} u^i \\ \nu_a + f_a^i u_i \end{pmatrix} \qquad \text{where} \qquad u_i = h_{ij} u^j ,$$

Non-holonomic Riemannian structure $(M, g, D, D^{\perp})$

Fix an extension of the metric to the entire TM. In particular, there is the orthogonal complement $D^{\perp}$.
Choose $E = TM$ and for $\alpha \geq 0$

$$\Phi_\alpha = \begin{cases} \mathrm{id}_D & \text{on } D \\ \alpha\, \mathrm{id}_{D^\perp} & \text{on } D^\perp. \end{cases}$$

When α approaches zero we charge each of the $D^{\perp}$ components of the velocities $\dot{c}(t)$ by a $1/\alpha$ multiple of its original size with respect to g. At the $\alpha = 0$ limit we obtain the original sub-Riemannian geometry.

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The Schouten connection

The projections of the Levi Civita connection to D and $D^{\perp}$ provide the Schouten connection ∇. This is a metric connection preserving $D \oplus D^{\perp}$.

The Hladky connection

Given a sub-Riemannian geometry (M, D, h), let g be a Riemannian metric on TM that restricts to h on D and write $D^\perp$ for the orthogonal complement of D. Then there is the unique metric connection[a] ∇ on TM such that both D and $D^\perp$ are preserved, and

$$T^{D}_{DD} = 0, \qquad T^{D^\perp}_{D^\perp D^\perp} = 0$$

$$T^{D^\perp}_{DD^\perp} \text{ is symmetric with respect to } g_{|D^\perp}$$

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This connection ∇ is invariant with respect to constant rescalings of g on D or $D^\perp$.

[a] R.K. Hladky, Connections and curvature in sub-Riemannian geometry. Houston J. Math. 38 (2012), no. 4, 1107-1134,
see also F. Baudoin, E. Grong, G. Molino, L. Rizzi, Comparison theorems on H-type sub-Riemannian manifolds, arXiv:1909.03532

Fix extension metric g of the given sub-Riemannian metric h, write $TM = D \oplus D^{\perp}$, consider the family of metrics $g^{\epsilon}_{|D} = g$ and $g^{\epsilon}_{|D^{\perp}} = \epsilon g$. They all share the Hladky connection ∇.
We rewrite the geodesic equation for the metric minimizers of g^{ϵ} in term of ∇ and its torsion.
Write D^{ϵ} for the Levi Civita connection of g^{ϵ} and $A^{\epsilon} : TM \otimes TM \to TM$ be the contorsion tensor,

$$D^{\epsilon}_X Y = \nabla_X Y + A^{\epsilon}(X, Y).$$

Consider local non-holonomic frames spanning D and $D^\perp$ and use indices $i, j, k, \dots$ and $a, b, c, \dots$ in relation to D and $D^\perp$, respectively, i.e., $u = u^i + u^a$ is the tangent curve $u = \dot{c}$, $\nabla = \nabla_i + \nabla_a$.
Similarly, write g_{ij} and ϵg_{ab}, and torsions

$$T^i{}_{jk} + T^i{}_{ja} + T^i{}_{ab} + T^a{}_{jk} + T^a{}_{jb} + T^a{}_{bc}.$$

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Observations

The symmetric parts of the torsions $T^i{}_{ja}$ and $T^a{}_{ib}$ are given by the formula (via polarization)

$$\langle X, T(Z,X)\rangle = \frac{1}{2} Z\|X\|^2 + \langle X, [X,Z]\rangle$$

The torsion $T^a{}_{ij}$ is just the Levi form $L^a{}_{ij}$, i.e. the projection of the Lie bracket of vector fields.

Variational equations for extremals

The variational equations $D^{\epsilon}_{u}u = 0$ for the tangent curves $u = \dot{c}^{\epsilon}$ of the g^{ϵ} critical curves c^{ϵ} are

$$\begin{aligned} 0 &= g_{ij}u^k\nabla_k u^j + g_{ij}u^a\nabla_a u^j + g_{kj}u^k T^j{}_{ia}u^a \\ &\qquad + \epsilon g_{ab}u^a T^b{}_{ic}u^c + \epsilon g_{ab}u^a T^b{}_{ik}u^k \\ 0 &= \epsilon g_{ab}u^k\nabla_k u^b + \epsilon g_{ab}u^c\nabla_c u^b + g_{ij}u^i T^j{}_{ab}u^b \\ &\qquad + g_{ij}u^i T^j{}_{ak}u^k + \epsilon g_{cb}u^b T^c{}_{ak}u^k. \end{aligned}$$

The renormalization

Next we "renormalize" the $D^{\perp}$ component u^a as

$$u^a = \frac{1}{\epsilon}\nu^a$$

and consider $\delta = 1/\epsilon$. In the limit $\delta = 0$ we arrive at

$$0 = g_{ij}u^k\nabla_k u^j + g_{ab}\nu^a T^b{}_{ik}u^k$$
$$0 = g_{ab}u^k\nabla_k \nu^b + g_{ij}u^i T^j{}_{ak}u^k + g_{cb}\nu^b T^c{}_{ak}u^k$$

With the help of g, we can view the result as equations coupling the components $(u^i) \in \mathcal{D}$ with (ν_a) in the annihilator of $\mathcal{D}$ in T^*M:

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Theorem

For each set of initial conditions $x \in M$, $u(0) \in \mathcal{D} \subset T_xM$, and $\nu(0) \in \mathcal{D}^\perp \subset T_x^*M$, the component $u(t)$ of the unique solution of the equations

$$
\begin{aligned}
0 &= u^k \nabla_k u^i + g^{ij} \nu_a L^a{}_{ik} u^k \\
0 &= u^k \nabla_k \nu_a + g_{ij} u^i T^j{}_{ak} u^k + \nu_b T^b{}_{ak} u^k
\end{aligned}
\tag{1}
$$

projects to a locally defined normal extremal $c(t)$ of the sub-Riemannian geometry with $c(0) = x$ and $\dot{c}(t) = u(t)$.

generalized Heisenberg in 5D

Holonomic coordinates (x^1, x^2, x^3, x^4, z), D spanned by

$$X_1 = \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial z} \qquad X_2 = \lambda\left(\frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial z}\right)$$

$$X_3 = \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial z} \qquad X_4 = \lambda\left(\frac{\partial}{\partial x^4} + x^2 \frac{\partial}{\partial z}\right)$$

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$$X_3 = \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial z} \qquad X_4 = \lambda\left(\frac{\partial}{\partial x^4} + x^2 \frac{\partial}{\partial z}\right)$$

The set of first 5 equations reads (here $u(t) = \alpha^i X_i$ in the non-holonomic frame)

$$\dot{x}^1 = \alpha^1, \qquad \dot{x}^2 = \lambda\alpha^2, \qquad \dot{x}^3 = \alpha^3, \qquad \dot{x}^4 = \lambda\alpha^4,$$
$$\dot{z} = x^1\alpha^3 - x^3\alpha^1 + \lambda x^2\alpha^4 - \lambda x^4\alpha^2,$$

Our "non-holonomic equations" then get

$$
\begin{aligned}
\dot\alpha^1 &= \frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4) - \nu\alpha^3,\\
\dot\alpha^2 &= -\frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}\alpha^1\alpha^2 - \frac{\lambda_{x^3} + x_1\lambda_z}{\lambda}\alpha^2\alpha^3 - (\lambda_{x^3} + x^2\lambda_z)\alpha^2\alpha^4\\
&\qquad - (x^4\lambda_z - \lambda_{x^2})\alpha^4\alpha^4 - \lambda^2\nu\alpha^4,\\
\dot\alpha^3 &= \frac{\lambda_{x^3} - x^1\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4) - \nu\alpha^1,\\
\dot\alpha^4 &= -\frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}\alpha^1\alpha^4 - \frac{\lambda_{x^3} + x_1\lambda_z}{\lambda}\alpha^3\alpha^4 + (\lambda_{x^4} + x^2\lambda_z)\alpha^2\alpha^2\\
&\qquad + (x^4\lambda_z - \lambda_{x^2})\alpha^2\alpha^4 + \lambda^2\nu\alpha^2,\\
\dot\nu &= \frac{2\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4).
\end{aligned}
$$

In particular, if $\lambda_z = 0$ then ν is a free constant parameter. These equations coincide with the standard ones if λ is a constant function.

Prolongation of sub-Riemannian geometries

Let $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$ be the Lie algebra of the Lie group G_0 of all automorphisms of the graded nilpotent algebra $\mathfrak{g}_-$ preserving the metric σ on $\mathfrak{g}_{-1}$.
The action of the derivations from $\mathfrak{g}_0$ on $\mathfrak{g}_-$ extends the Lie algebra structure on $\mathfrak{g}_-$ to the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

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Observation 1

The Tanaka prolongation of $\mathfrak{g}$ is finite.[a]

[a] Corollary 2 of Theorem 11.1 in *Tanaka, N.*, On differential systems, graded Lie algebras and pseudo-groups, *J. Math. Koyto Univ., 10, 1 (1970), 1-82.*

Observation 2

Already the first prolongation is trivial.[a] Thus $\mathfrak{g}$ is the full prolongation of $\mathfrak{g}_-$.

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Theorem

For each subriemannian manifold (M, D, S) *with constant symbol, there is the unique Cartan connection* $(\mathcal{G} \to M, \omega)$ *of type* $(\mathfrak{g}, G_0)$ *with the curvature function* $\kappa : \mathcal{G} \to \mathfrak{g} \otimes \Lambda^2 \mathfrak{g}_-^*$ *satisfying* $\partial^* \kappa = 0$*. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection* κ_H*, i.e. the component with* $\partial \kappa_H = 0$ *as well.*[a]

[a] Morimoto, T., *Cartan connection associated with a subriemannian structure*, Differential Geometry and its Applications 26 (2008), 75-78.

Underlying filtered geometry

The distribution D on M itself often defines a nice finite type filtered geometry which enjoys a canonical Cartan connection, too. Many of them belong to the class of the parabolic geometries, for which the full Tanaka prolongation of $\mathfrak{g}_-$ is a semisimple Lie algebra

$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

and $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is the opposite nilpotent radical to the parabolic subalgebra $\mathfrak{p} = \bar{\mathfrak{g}}_0 \oplus \cdots \oplus \bar{\mathfrak{g}}_k \subset \bar{\mathfrak{g}}$, with $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$.

Fix one such graded semisimple $\bar{g}$ and consider the frame bundle $\mathcal{G}_0 \to M$ of gr TM giving a parabolic geometry. Often the structure group G_0 of $\mathcal{G}_0$ is the full group of graded automorphisms of $\mathfrak{g}_-$.[1] Adding a metric S on D, we have got two (related) Cartan connections there.

[1] See Čap, A., Slovák, J., Parabolic Geometries I, Background and General Theory, AMS, Math. Surveys and Monographs 154, x+628pp. for details.

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Theorem

Consider a bracket generating distribution D on M with the constant symbol equal to the negative part of a graded semisimple Lie algebra $\bar{\mathfrak{g}}$ and the corresponding frame bundle $\mathcal{G}_0 \to M$ of gr TM. Then there is the unique Cartan connection $(\bar{\mathcal{G}} \to M, \omega)$ of type $(\bar{\mathfrak{g}}, P)$ with the curvature function $\bar{\kappa} : \bar{\mathcal{G}} \to \bar{\mathfrak{g}} \otimes \Lambda^2 \mathfrak{g}_-^*$ satisfying $\partial^* \bar{\kappa} = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection $\bar{\kappa}_H$, i.e. the component with $\partial \bar{\kappa}_H = 0$ as well.

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Consider a parabolic geometry (M, D) equipped by the metric S on D, assume (M, D, S) has got constant symbol.
Thus we have got:

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$
$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

This is an instance of a $\mathfrak{g}_-$-submodule W of $\mathfrak{g}_-$-module V.

The short exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

induces the short exact sequence of differential complexes

$$0 \longrightarrow C^\bullet(\mathfrak{g}_-, W) \xrightarrow{i} C^\bullet(\mathfrak{g}_-, V) \xrightarrow{\pi} C^\bullet(\mathfrak{g}_-, V/W) \longrightarrow 0$$

and thus the long exact sequence in cohomologies

$$\longrightarrow H^n(\mathfrak{g}_-, W) \xrightarrow{i} H^n(\mathfrak{g}_-, V) \xrightarrow{\pi} H^n(\mathfrak{g}_-, V/W) \xrightarrow{\delta} H^{n+1}(\mathfrak{g}_-, W) \xrightarrow{i} H^{n+1}(\mathfrak{g}_-, V) \xrightarrow{\pi} H^{n+1}(\mathfrak{g}_-, V/W) \longrightarrow$$

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Notice, the filtration is induced by the distribution D and we declare its symbol to be equal to the Lie algebra $\mathfrak{g}_-$ at all points. Thus, all the curvature homogeneities ≤ 0 vanish.

Theorem

Let $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_- \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_+$ be a graded Lie algebra and $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$ be a non-positively graded Lie algebra such that $\mathfrak{g}_- = \bar{\mathfrak{g}}_-$ and $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$. The cohomology $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ as a $\mathfrak{g}_0$-submodule is isomorphic to a direct sum of 2 parts:

1. $H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})/H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$,
2. $\ker \pi \subset H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$,

where $\pi\colon H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ is the projection induced by the projection in cochains $\pi\colon C^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to C^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$.

Recall, many generic bracket generating distributions $\mathcal{H} \subset TM$ define a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P).
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sub-Riemannian metrizability

With G semisimple and $P \subset G$ parabolic, there is the class $[\nabla_{|\mathcal{H}}]$ of partial Weyl connections on M and we may look for metrics on $\mathcal{H}$ parallel in the $\mathcal{H}$-directions for at least one of the connections in the class.

The homogeneous model

On $M = G/P$, the exponential coordinates identify the big cell $\exp \mathfrak{g}_- \subset M$ with the nilpotent group.
This identification yields the reduction of $G \to G/P$ to the Levi factor P_0 of P and thus the (very flat) Weyl connection ∇ (as a reduction of the Maurer Cartan form ω).
Any metric on $\mathfrak{h} \subset T_O M$ in the origin can be uniquely extended by left shifts and this provides a parallel metric on the big cell.
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This is the usual *nilpotent approximation* in the geometric control theory.
Our aim is to consider the G_0-irreducible components of the metrics on $\mathfrak{h}$ with G-dominant highest weights. Let us write W for those tractors. It turns out that under certain conditions on the weights (limited number of components in certain tensor products), the space of the metrics admitting parallel Weyl connections is in 1-1 correspondence with the parallel tractors.

The metric tractor bundle and first BGG operator

Consider $\mathcal{V}$, a bundle coming from a G representation $\mathbb{V}$.
There is the so called **BGG sequence of invariant operators**. All the bundles here come from P-modules with $\mathfrak{p}$-dominant highest weights on the same affine orbit of the Weyl group as $\mathbb{V}$.

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The solutions are always expressed by **polynomial formulae** in the flat normal coordinates at the homogenous model.
Still true for those solutions on general curved geometries which are determined by parallel sections of the corresponding tractor bundle, the so called **normal solutions**.

The free distribution case

Here $\mathfrak{g} = \mathfrak{so}(n+1, n) = \Lambda^2\mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{h}^* \oplus \Lambda^2\mathfrak{h}^*$, $\mathcal{B} = S^2\mathfrak{h}$ is irreducible and satisfies the ALC.

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The *standard tractor bundle* is the bundle associated to the defining representation V of $G = SO(n+1,n)$. Explicitly, $v = (\lambda^a, \tau, \ell_a)^T$ with the action

$$\begin{pmatrix} 0 & x^a & y^{ab} \\ 0 & 0 & -x^a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^b \\ \tau \\ \ell_b \end{pmatrix} = \begin{pmatrix} x^a\tau + y^{ab}\ell_b \\ -x^b\ell_b \\ 0 \end{pmatrix}.$$

The metric tractor bundle in this example is associated to the symmetric tracefree square $S^2_0 V$ of V.

The free distribution case

Here $\mathfrak{g} = \mathfrak{so}(n+1, n) = \Lambda^2\mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{h}^* \oplus \Lambda^2\mathfrak{h}^*$, $\mathcal{B} = S^2\mathfrak{h}$ is irreducible and satisfies the ALC.

The *standard tractor bundle* is the bundle associated to the defining representation V of $G = SO(n+1, n)$. Explicitly, $v = (\lambda^a, \tau, \ell_a)^T$ with the action

$$\begin{pmatrix} 0 & x^a & y^{ab} \\ 0 & 0 & -x^a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^b \\ \tau \\ \ell_b \end{pmatrix} = \begin{pmatrix} x^a\tau + y^{ab}\ell_b \\ -x^b\ell_b \\ 0 \end{pmatrix}.$$

The metric tractor bundle in this example is associated to the symmetric tracefree square $S^2_0 V$ of V.

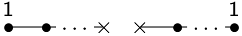
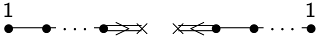
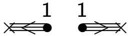
The normal solutions are (in the normal coordinates) given by

$$\eta^{ab}(x,y) = \nu^{ab} + x^{(a}\sigma^{b)} - y^{c(a}\psi_c^{b)} + \tfrac{1}{2}x^c x^{(a}\psi_c^{b)} + x^{(a}y^{b)c}\xi_c + \tfrac{1}{2}x^a x^b \psi_c^c + \tfrac{2}{3}x^a x^b x^c \xi_c - \tfrac{1}{3}x^{(a}y^{b)c}x^d\tau_{dc} - \tfrac{1}{6}x^a x^b x^c x^d \tau_{cd}.$$

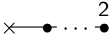
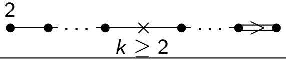
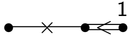
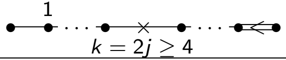
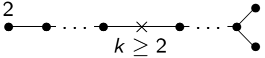
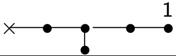
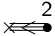
The following tables classify triples $\mathfrak{p}, \mathfrak{g}, B$, where ∂ is a parabolic subalgebra in a real simple Lie algebra $\mathfrak{g}$, and B is an irreducible $\mathfrak{p}$-submodule of $S^2\mathfrak{h}$, with $\mathfrak{h} \cong (\mathfrak{p}^\perp/[\mathfrak{p}^\perp, \mathfrak{p}^\perp])^*$ irreducible, satisfying the ALC and admiting nondegenerate elements.
There are 14 infinite series of geometries and 6 special cases.

The following tables classify triples $\mathfrak{p}, \mathfrak{g}, B$, where ∂ is a parabolic subalgebra in a real simple Lie algebra $\mathfrak{g}$, and B is an irreducible $\mathfrak{p}$-submodule of $S^2\mathfrak{h}$, with $\mathfrak{h} \cong (\mathfrak{p}^\perp/[\mathfrak{p}^\perp, \mathfrak{p}^\perp])^*$ irreducible, satisfying the ALC and admiting nondegenerate elements.
There are 14 infinite series of geometries and 6 special cases.

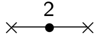
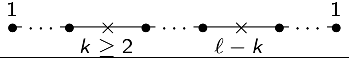
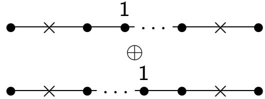
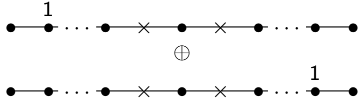
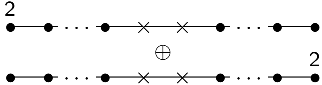
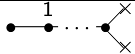
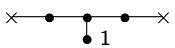
Complex geometries with hermitian B

Case	Diagram Δ_ℓ for $\mathfrak{p}, B$	Real simple $\mathfrak{g}$	Growth
A_ℓ^h		$\mathfrak{sl}(\ell+1, \mathbb{C})$ $\ell \geq 2$	2ℓ
B_ℓ^h		$\mathfrak{so}(2\ell+1, \mathbb{C})$ $\ell \geq 2$	$2k, 2k+k(k-1)$
G_2^h		$G_2^{\mathbb{C}}$	$4, 6, 10$

Real geometries with absolutely irreducible $\mathfrak{h}$

Case	Diagram Δ_ℓ for $\mathfrak{p}, B$	Real simple $\mathfrak{g}$	Growth
$A_\ell^{1,1}$		$\mathfrak{sl}(\ell+1,\mathbb{R})\ \ \ell \geq 2$	ℓ
$A_\ell^{1,2}$		$\mathfrak{sl}(\ell+1,\mathbb{R}),\ \mathfrak{sl}(p+1,\mathbb{H})$ $\ell = 2p+1,\ p \geq 2$	$4p$
$B_\ell^{1,k}$		$\mathfrak{so}(p,q),\ \ k \leq p \leq q$ $p+q = 2\ell+1$	$d = k(2\ell-2k+1),$ $n = d + \frac{1}{2}k(k-1)$
$B_\ell^{1,\ell}$		$\mathfrak{so}(\ell,\ell+1)\ \ell \geq 2$	$k, k+\frac{1}{2}k(k-1)$
$C_4^{1,2}$		$\mathfrak{sp}(8,\mathbb{R})$ $\mathfrak{sp}(2,2)\ \ \mathfrak{sp}(1,3)$	$8, 11$
$C_\ell^{1,k}$		$\mathfrak{sp}(2\ell,\mathbb{R})\ \ \ \mathfrak{sp}(p,q)$ $\ell = p+q,\ \ k \leq p \leq q$	$d = k(2\ell-2k),$ $n = d + \frac{1}{2}k(k+1)$
$D_\ell^{1,k}$		$\mathfrak{so}(p,q)\ \ \ \ \ \mathfrak{so}^*(2\ell)$ $2\ell = p+q\ \ \ \ k = 2j$ $k \leq p \leq q\ \ \ \ k \leq \ell-2$	$d = k(2\ell-2k),$ $n = d + \frac{1}{2}k(k-1)$
$E_6^{1,1}$		$E_{6(6)},\ E_{6(-26)}$	16
$G_2^{1,1}$		$G_{2(2)}$	$2, 3, 5$

Real geometries with $\mathfrak{h}$ not absolutely irreducible

Case	Diagram Δ_ℓ for $\mathfrak{p}, B$	Real simple $\mathfrak{g}$	Growth
$A_3^{2,1}$		$\mathfrak{su}(1,3),\ \mathfrak{su}(2,2)$	$4, 5$
$A_\ell^{2,k}$		$\mathfrak{su}(p,q),\ k \leq p \leq q$ $\ell = p + q - 1 \geq 4$	$d = 2k(\ell - 2k + 1$ $n = d + k^2$
$A_\ell^{2,h}$		$\mathfrak{su}(p,q),\ 2 \leq p \leq q$ $\ell = p + q - 1 \geq 6$	$4(\ell - 3), 4(\ell - 2)$
$A_{2k+1}^{2,s}$		$\mathfrak{su}(k, k+2)$, $\mathfrak{su}(k+1, k+1)$ $\ell = 2k + 1 \geq 7$	$4k, 4k + k^2$
$A_{2k}^{2,s}$		$\mathfrak{su}(k, k+1)$ $\ell = 2k \geq 4$	$2k, 2k + k^2$
$D_\ell^{2,s}$		$\mathfrak{so}(\ell - 1, \ell + 1)$ $\mathfrak{so}^*(2\ell),\ \ell = 2j + 1$	$d = 2(\ell - 1)$, $d + \frac{1}{2}(\ell - 1)(\ell -$
$D_\ell^{2,h}$		$\mathfrak{so}(\ell - 1, \ell + 1)$ $\mathfrak{so}^*(2\ell),\ \ell = 2j + 1$	$d = 2(\ell - 1)$, $d + \frac{1}{2}(\ell - 1)(\ell -$
$E_6^{2,h}$		$E_{6(2)}$	$16, 24$

HOPE TO SEE SOME MORE EXAMPLES AND APPLICATIONS !!

Thank you for attention!