Beyond treewidth: the tree-independence number

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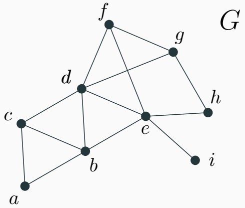
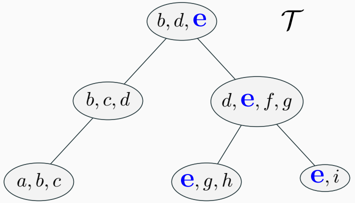
Treewidth of a graph

The **treewidth** of a graph G, denoted by $\mathrm{tw}(G)$, measures how similar the graph is to a tree.

A **tree decomposition** of a graph G is a collection $\mathcal{T}$ of **bags** (subsets of $V(G)$) arranged into a tree T such that

- each vertex of G is in a bag,
- for each edge of G, both endpoints are in a bag, and
- for each vertex v of G, the nodes of T whose bags contain v form a subtree.





The **width** of $\mathcal{T}$ is the size of the largest bag minus one.
The **treewidth** of G is the minimum width over all possible tree decompositions.

Many origins of treewidth:

- 1972: Bertelè and Brioschi (dimension)
- 1976: Halin (S-functions of graphs)
- 1984: Robertson and Seymour (treewidth)
- 1985: Arnborg and Proskurowski (partial k-trees)

Many uses of treewidth:

- **Structural graph theory**
- **Algorithms**
- **Logic**
- **General**: Many application areas: artificial intelligence, statistical machine learning, Bayesian networks, databases, social networks, programming languages, etc.

Tree decompositions are a great tool for **dynamic programming**.

In particular, this is the case for Maximum Weight Independent Set:

- We are given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Q}_+$.
- The task is to find an independent set I in G of maximum possible weight $w(I)$, where

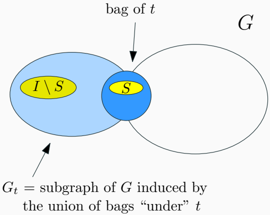
$$w(I) = \sum_{x \in I} w(x)\,.$$

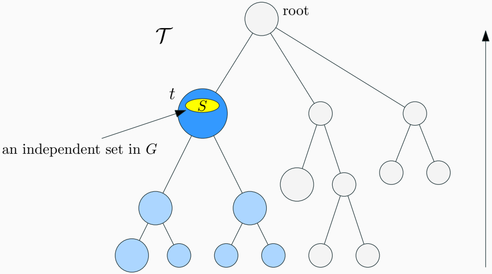
Independent set: a set of pairwise non-adjacent vertices

Dynamic programming for MAXIMUM WEIGHT INDEPENDENT SET:

Root the tree decomposition and traverse it bottom-up.

For each node t and each independent set S contained in the bag of t, we compute the maximum weight of an independent set I in G_t that agrees with S in the bag.



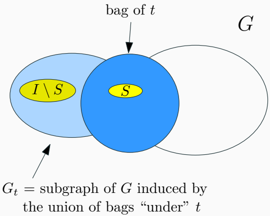


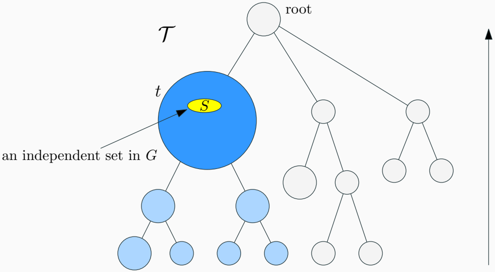
This computation can be done recursively, using separation properties of tree decompositions. If each bag has at most a constant number k of vertices, we only need to examine $\leq 2^k$ choices for S.

This idea can be generalized:

Suppose that in each bag we only need to examine **a polynomial number** of choices for S, say $\mathcal{O}(n^k)$ where $n = |V(G)|$ and k is a constant

- Bags may be large, but each independent set contained in a bag is small.





If G is given with such a tree decomposition, we still obtain a polynomial-time algorithm for Maximum Weight Independent Set.

Tree-independence number: the definition

The **independence number** of a graph G, denoted by $\alpha(G)$, is the maximum size of a independent set in G.

The **independence number** of a tree decomposition of a graph G is the maximum, over all bags of the decomposition, of the independence number of the subgraph of G induced by the bag.

The **tree-independence number** of G is the minimum independence number over all tree decompositions.

Notation: tree-$\alpha(G)$

Similar invariants were studied in the literature with respect to various properties of the bags:

- connectivity properties (connected treewidth, Diestel and Müller, 2018),
- metric properties (tree-length, Dourisboure and Gavoille, 2007, tree-breadth, Dragan and Köhler, 2014),
- chromatic properties (tree-chromatic number, Seymour, 2016).

Examples of graph classes of bounded tree-independence number:

- Graph classes of bounded treewidth:

$$\text{tree-}\alpha(G) \leq \mathrm{tw}(G) + 1\,.$$

- Graph classes of bounded independence number:

$$\text{tree-}\alpha(G) \leq \alpha(G)\,.$$

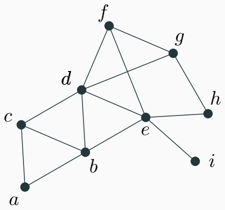
- Intersection graphs of connected subgraphs of graphs with treewidth t (Scheffler, 1990; Bodlaender, Gustedt, Telle, 1998)

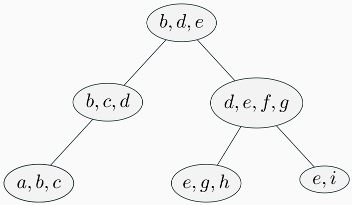
$$\text{tree-}\alpha(G) \leq t\,.$$

  - This includes chordal graphs ($t = 1$) and circular-arc graphs ($t = 2$).

- Classes of graphs in which all minimal separators are of bounded size (Skodinis, 1999) tree-$\alpha(G) \leq 1$: G has a tree decomposition in which each bag is a clique

This happens if and only if G is chordal.





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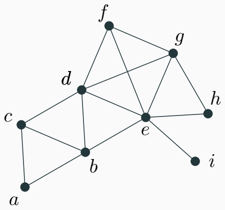

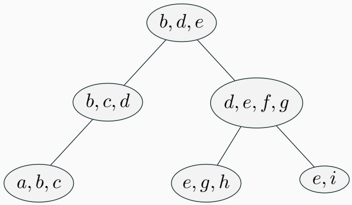

Computing the tree-independence number of a given graph is **NP-hard**.

- We reduce from the INDEPENDENT SET problem: given a graph G, the graph G' obtained from two copies of G by adding all edges between them satisfies

$$\text{tree-}\alpha(G') = \alpha(G)\,.$$

In particular, tree-$\alpha(K_{n,n}) = n$.

Our main result

We consider **six graph containment relations** (the subgraph, topological minor, and minor relations, as well as their induced variants).

For each of them we completely characterize
graph classes of bounded tree-independence number
defined by a single forbidden graph with respect to the relation.

For each of the obtained bounded cases, we show that a tree decomposition with small independence number can be computed efficiently.

This leads to new graph classes in which
MAXIMUM WEIGHT INDEPENDENT SET can be solved in polynomial time.

- This includes an infinite family of generalizations of the class of **chordal graphs**, for which a polynomial-time algorithm for the MWIS problem was given by Frank in 1976.

Examples of containments:

Six graph containment relations

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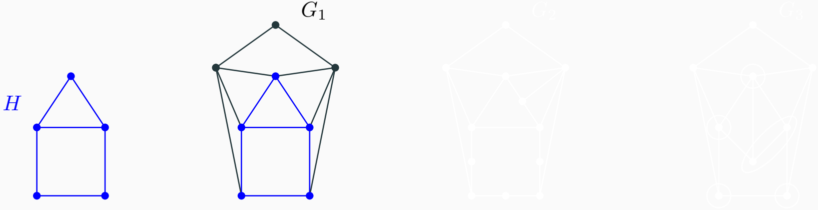
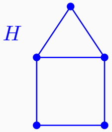
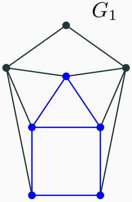


$H \subseteq_{is} G_1$

induced
subgraph

Examples of containments:





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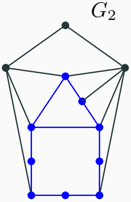
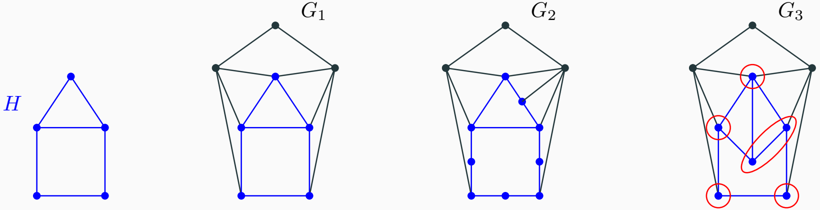


$H \subseteq_{itm} G_2$

induced
topological minor

Six graph containment relations

Examples of containments:



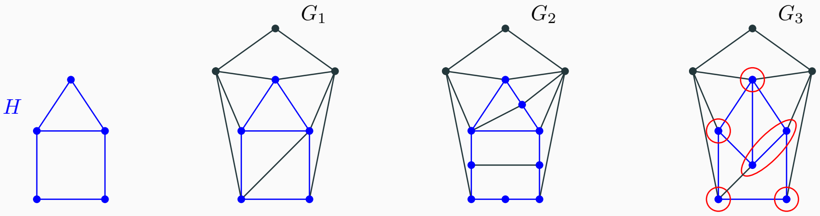
$H \subseteq_{is} G_1$ — induced subgraph

$H \subseteq_{itm} G_2$ — induced topological minor

$H \subseteq_{im} G_3$ — induced minor

Six graph containment relations

Examples of containments:



$H \subseteq_s G_1$	$H \subseteq_{tm} G_2$	$H \subseteq_m G_3$
subgraph	topological minor	minor

Summary of characterizations

Six dichotomy theorems

Graphs H for which the class of graphs excluding H has bounded tree-independence number:

	Non-induced	Induced
Subgraph	$H \in \mathcal{S}$	P_3 or edgeless
Topological minor	H is subcubic and planar	C_4, K_4^-, edgeless
Minor	H is planar	W_4, K_5^-, $K_{2,q}$ for some $q \in \mathbb{N}$

$\mathcal{S}$ is the class of graphs whose connected components are either paths or subdivisions of the claw ($K_{1,3}$).

That these are conditions are **necessary** follows from our previous work, where for each relation we completely characterized graphs H for which the class of graphs excluding H (wrt the relation) is (tw, ω)-bounded.

A graph class $\mathcal{G}$ is **(tw, ω)-bounded** if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ and every induced subgraph G' of G, we have $\mathrm{tw}(G') \leq f(\omega(G'))$.

The **clique number** of a graph G, denoted by $\omega(G)$, is the maximum size of a clique in G.

Ramsey's Theorem implies that bounded tree-independence number is a sufficient condition for (tw, ω)-boundedness.

Excluding a single **subgraph, topological minor, or minor**:
bounded tree-independence number is equivalent to **bounded treewidth**.

Excluding a single **induced subgraph**:
bounded tree-independence number is equivalent to one of the following:

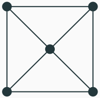
- the class has **bounded independence number**,
- every connected graph in the class is complete.

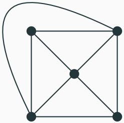
Excluding a single **induced topological minor**:
bounded tree-independence number is equivalent to one of the following:

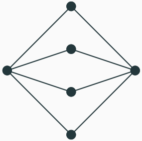
- the class has **bounded independence number**,
- the class is a subclass of the class of **chordal graphs** (tree-$\alpha \leq 1$),
- the class is a subclass of the class of **block-cactus graphs** (every block is either complete or a cycle; tree-$\alpha \leq 2$).

The most interesting case: induced minors

Now, let H be excluded wrt the **induced minor** relation.


4-wheel


K_5^-


$K_{2,4}$

We have two main cases:

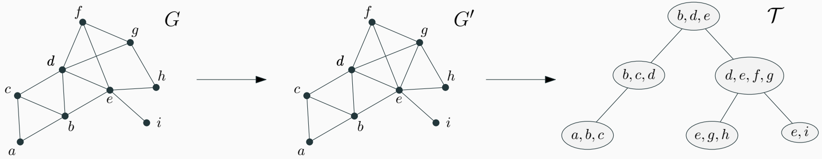
H is the 4-wheel or K_5^-:	tree-α(H-induced-minor-free graphs) ≤ 4
$H = K_{2,q}$ for some $q \geq 2$:	tree-α(H-induced-minor-free graphs) $\leq 2(q-1)$

Proof idea for $H = K_{2,q}$ for $q \geq 2$

Now let $\mathcal{G}$ be the class of $K_{2,q}$-induced-minor-free graphs, $G \in \mathcal{G}$.

1. We compute a **minimal triangulation** G' of G (= we add edges to G in a minimal way to make it chordal).
   - Can be done in time $\mathcal{O}(|V(G)|^{\mu} \log |V(G)|)$, where $\mu < 2.37286$ is the matrix multiplication exponent (Heggernes, Telle, and Villanger, 2005).

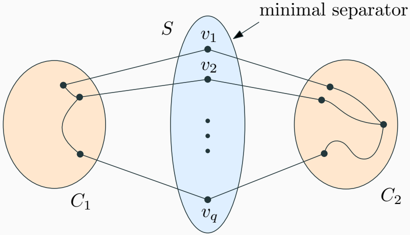
2. We compute a **clique tree** of G' = a tree decomposition of G' having exactly the maximal cliques of G' as bags
   - Can be done in time $\mathcal{O}(|V(G')| + |E(G')|)$ (e.g., Berry and Simonet, 2017)



3. A clique tree of G' is also a tree decomposition of G.

   Following the approach of Bouchitté and Todinca, 2002,
   we show that each bag (which is a **potential maximal clique** of G)
   is either a clique in G or can be covered by two minimal separators.

4. **Observation:** Since G is $K_{2,q}$-induced-minor-free, each minimal separator induces a subgraph with independence number less than q.



   We thus obtain a tree decomposition with independence number at most $2(q-1)$.

Proof idea for $H \in \{W_4, K_5^-\}$

Fix $H \in \{W_4, K_5^-\}$ and let $\mathcal{G}$ be the class of H-induced-minor-free graphs.

1. We characterize the **3-connected graphs** in $\mathcal{G}$.
    - $H = W_4$: (3-connected) chordal graphs
    - $H = K_5^-$: complete graphs K_n with $n \geq 4$, wheels, $K_{3,3}$, $\overline{C_6}$

The characterizations lead to **quickly computable tree decompositions with small independence number** (and satisfying some additional technical conditions).

2. We **reduce the problem to the 3-connected case**.
   This is done in two steps:

   - From general to 2-connected graphs using the **block-cutpoint tree** (computable in linear time using DFS, Hopcroft and Tarjan, 1973).
   - From 2-connected to 3-connected graphs using the **SPQR tree** (computable in linear time, Hopcroft and Tarjan, 1973, Gutwenger and Mutzel 2001, Dujmović, Eppstein, Joret, Morin, Wood, 2020).

We show that the tree decompositions of the triconnected components of a graph $G \in \mathcal{G}$ can be efficiently combined into a tree decomposition of the whole graph, while (roughly) preserving the independence number.

- A useful fact: each triconnected component of a graph G is an **induced topological minor of G**.

Algorithmic consequences

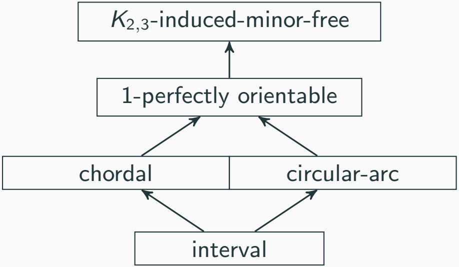
Maximum Weight Independent Set is solvable in:

- time $\mathcal{O}(|V(G)|^6)$ for W_4-induced-minor-free graphs,
- time $\mathcal{O}(|V(G)|^6)$ for K_5^--induced-minor-free graphs,
- time $\mathcal{O}(|V(G)|^{2q})$ for $K_{2,q}$-induced-minor-free graphs, for all $q \geq 2$.

The algorithms are **robust**: we do not need to know if the input graph belongs to the class.

- If it does not, the algorithm either correctly solves the problem or correctly detects that the graph is not in the class.

Corollary: Maximum Weight Independent Set is solvable in time $\mathcal{O}(|V(G)|^6)$ in the class of 1-perfectly orientable graphs.



This answers an open question of Beisegel, Chudnovsky, Gurvich, M., and Servatius (2019).

Open questions

Question 1:
Is MAXIMUM WEIGHT INDEPENDENT SET solvable in polynomial time in any hereditary class of graphs with bounded tree-independence number?

Question 2:
For fixed $k \geq 2$, what is the complexity of determining if the tree-independence number of a given graph is at most k?

Question 3:
Does every (tw, ω)-bounded graph class have bounded tree-independence number?

Thank you!

Questions?