The Calderón problem with corrupted data

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8th European Congress of Mathematics
Analysis, Control and Inverse Problems for PDE
23 June 2020

Outline

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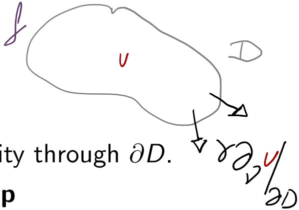
The inverse Calderón problem

The **inverse Calderón problem** aims at determining the conductivity of an inhomogeneous conductive medium from non-invasive measurements.

The formulation

If f is an **electric potential** prescribed on ∂D, the **electric potential** u inside of D satisfies

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } D, \\ u|_{\partial D} = f. \end{cases}$$



- $\gamma \partial_\nu u|_{\partial D}$ is the **outgoing electric current** density through ∂D.
- Measurements: the **Dirichlet-to-Neumann map**

$$\Lambda_\gamma : f \longmapsto \gamma \partial_\nu u|_{\partial D}$$

The **inverse Calderón problem** is

- to decide if γ is uniquely determined by Λ_γ,
- and to calculate γ whenever there is unique determination.

Discussing the model

This problem originates as a theoretical model in electrical prospecting.

- The aim is to determine the conductivity conductivity by means of steady state electrical measurements on ∂D.

Ideally, Λ_γ is determined through measurements effected on ∂D.
The model assumes to have access (to the graph of the DN map):

- to infinite many pieces of data
- and to infinite-precision measurements.

This is unjustified (data do not lie on the graph of the DN map):

- only a finite number of measurements are available
- the data is corrupted by measurement errors

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Boundary reconstruction (smooth setting)

In 1988 Sylvester–Uhlmann show that, whenever D and γ are smooth, the DN map Λ_γ can be locally identified with a first order pseudodifferential operator, and its symbol can be expanded as:

$$\sum_{j=0}^{\infty} \partial_\nu^j \gamma.$$

Then, to reconstruct $\partial_\nu^j \gamma|_{\partial D}$ we only need to recover the symbol of a pseudodifferential operator —this is well known.

$$P = \sum_{|\alpha|\leq m} a_\alpha(x)(-i)^{|\alpha|}\partial_x^\alpha \quad \Rightarrow \quad e^{-ix\cdot\xi}P(e^{ix\cdot\xi}) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$$

The **plane waves** $e^{ix\cdot\xi}$ are the tools.

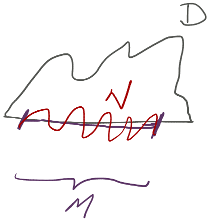
Boundary reconstruction (non-regular setting)

In 2001 Brown used solutions with highly oscillatory Dirichlet data concentrating around a point $P \in \partial D$ to recover

$$\gamma(P).$$

To visualize these solutions think of $P = 0 \in \partial D$ and

$$D \subset \{x \in \mathbb{R}^d : x_d > 0\}.$$



Then, the Dirichlet data of the solutions looks like

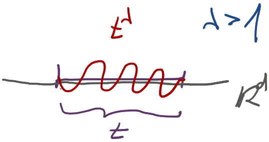
$$M^{(d-1)/2} N^{-1/2} \chi(Mx) e^{N(i\xi - e_d)\cdot x}|_{x_d=0}.$$

In 2006 Brown–Salo extended the method to recover:

$$\partial_\nu \gamma(P).$$

The tools are **wave packets**

$$f_{t,\lambda}(x) = t^{d/2} \chi(t(x - x_0)) e^{it^\lambda (x - x_0)\cdot \xi_0}.$$



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The Calderón problem with noisy data

Recall that the goal is to reconstruct γ from the DN map

$$\int_{\partial D} \Lambda_\gamma f\, g \;=\; \int_{\partial D} \gamma \partial_\nu u\, g$$

where

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } D, \\ u|_{\partial D} = f. \end{cases}$$

In order to avoid the infinite-precision assumption of Calderón's formulation, we assume data to be the DN map plus a random error:

$$\mathcal{N}_\gamma(f,g) = \int_{\partial D} \Lambda_\gamma f\, g \;+ \mathcal{E}(f,g),$$

where we want $\mathcal{E}(f,g)$ to denote a centred complex Gaussian whose variance depends on f and g.

Comments on the expectation

Note that

$$\mathbb{E}\mathcal{N}_\gamma(f,g) = \int_{\partial D} \Lambda_\gamma f\, g.$$

Therefore, the noise can be filtered having access to many independent outcomes:

$$\frac{1}{N}\sum_{n=1}^{N} \mathcal{N}_\gamma(f,g)_{(\omega_n)} \xrightarrow[N\to\infty]{} \int_{\partial D} \Lambda_\gamma f\, g.$$

- A few repetitions of the same measurement do not oscillate enough to filter out the noise by averaging.
- We want to avoid averaging and show that a single realization of $\mathcal{N}_\gamma$ is enough to reconstruct γ.

Comments on the variance

The variance

$$\mathbb{E}\left|\mathcal{N}_\gamma(f,g) - \int_{\partial D} \Lambda_\gamma f\, g\right|^2$$

depends on f and g.

- The *variable* variance aims at modelling measurement devices which decalibrates as the *strength* of the electric potential and the outgoing current increases.

Different approaches

There seem to be two different approaches.

- Deterministic regularization [Tikhonov]: The noise is deterministic and small.
- Statistical point of view: [Sudakov–Halfin, Franklin] No smallness assumption for the noise. [Abraham–Nickl] The level of noise is small.

Our approach is stochastic with no restriction on the size of the noise. We do not know a similar approach for the Calderón problem.

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Reconstruction of $\gamma|_{\partial D}$

Wave packets

Theorem (C, Garcia)

For every $P \in \partial D$, there exists an explicit sequence $\{f_N : N \in \mathbb{N}\}$ such that

$$\lim_{N\to\infty} \mathcal{N}_\gamma(f_N, \overline{f_N}) = \gamma(P)$$

almost surely.

Reconstruction of $\partial_\nu \gamma|_{\partial D}$

The DN map of the reference medium with conductivity identically one is denoted by Λ.

Theorem (C, Garcia)

wave packets

For every $P \in \partial D$, there exists an explicit family $\{f_t : t \geq 1\}$ such that, if

$$Y_N = \frac{1}{N^4} \int_{N^4}^{2N^4} \left[\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) - \int_{\partial D} \Lambda f_{t^2}\, \overline{f_{t^2}} \right] dt,$$

one has that

$$\lim_{N\to\infty} Y_N = \frac{\partial_{\nu_P}\gamma(P) + i\tau_P \cdot \nabla\gamma(P)}{\gamma(P)}$$

almost surely. Here ν_P is the outward unit normal vector to ∂D at P and τ_P denotes any unitary tangential vector at P.

Why the need of Y_N?

Recall that Λ_γ is a first order pseudodifferential operator with symbol

$$\sum_{j=0}^{\infty} \partial_\nu^j \gamma.$$

Since $\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}) \to \gamma(P)$, then

$$\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) \to 1.$$

Consequently,

$$\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) - \int_{\partial D} \Lambda f_{t^2}\, \overline{f_{t^2}} \to \frac{\partial_{\nu_P}\gamma(P)}{\gamma(P)}.$$

Comments on the results

- We also established a *high probable* rate of convergence.
- These results have been extended to Maxwell's equations in collaboration with Lai, Lin and Zhou.
- The main contributions are to filter the measurement errors. The underlying idea is the strong law of large numbers.
- For the first theorem no average is needed because

$$\|f_N\|_{L^2(\partial D)} = \mathcal{O}(N^{-1/2}).$$

- For the second theorem we require an average in $\sqrt{N}$ since

$$\|f_N\|_{L^2(\partial D)} = \mathcal{O}(1).$$

- An natural question: Could we recover $\partial_\nu^j \gamma|_{\partial D}$ for all $j \in \mathbb{N}_0$?

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The problem of observability with noise

The problem of recovering an **observable** P from certain **measurements** $\mathcal{N}_P$ that contain some random errors.

- The observable P is a pseudodifferntial operator

$$Pf(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi) \widehat{f}(\xi)\, \mathrm{d}\xi,$$

with a classical symbol a of order $m \in \mathbb{R}$.

- The measurements

$$\mathcal{N}_P(f,g) = \int_{\mathbb{R}^d} \overline{f} Pg + \mathcal{E}(\overline{f}, g).$$

error

observable

The observational limit of wave packets

Assume

$$a \sim \sum_{j=1}^{\infty} a_j,$$

with a_j being a classical symbol of order $m_j \in \mathbb{R}$, for $m_j < m_{j-1} < \cdots < m_1 = m$, which is *homogeneous in the variable* ξ.
In collaboration with Meroño, we showed how to use wave packets to reconstruct

$$a_1, \ldots, a_{j_0}, a_{j_0+1}, \ldots, a_{k_0}$$

when the observable P is so that

$$m = m_1 > \cdots > m_{j_0} > 0 \geq m_{j_0} > \cdots > m_{k_0} > -1/2.$$

- For $a_1, \ldots, a_{j_0}$ no averaging is needed.
- For $a_{j_0+1}, \ldots, a_{k_0}$ averaging is required.

Furthermore, it is not possible to use wave packets to reconstruct

$$a_{k_0+1}, a_{k_0+2}, \ldots$$

in presence of the error. The signal is lost in the noise.

A particular case to keep in mind

- If the observable P is a differential operator of order m, we can recover the full operator P from $\mathcal{N}_P$.
- Using wave packets, it is impossible to recover $\partial_\nu^j \gamma|_{\partial D}$ a.s. for $j \geq 2$.

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To sum up

- The Calderón problem is a theoretical model that arises in electrical prospecting.
- Implementing the model presents non-trivial challenges since it assumes infinite-precision measurements and infinite many pieces of data.
- We consider the problem of data corruption in the boundary reconstruction. Our approach is stochastic and provides reconstruction almost surely.
- Wave packets are useful but have limitations in the problem of observability with noise.