A generalization of a local form of the classical Markov inequality

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Classical Markov inequality

Markov's inequality

$$\|P'\|_{[a,b]} \leq \frac{2}{b-a}(\deg P)^2 \|P\|_{[a,b]},$$

where $\|f\|_K := \sup_{x \in K} |f(x)|$.

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Local form of the classical Markov inequality

$$|P'(x)| \leq \frac{1}{\epsilon}(\deg P)^2 \|P\|_{[x-\epsilon,x+\epsilon]}, \quad x \in \mathbb{R},\ \epsilon > 0.$$

A generalization to several variables of the classical Markov inequality

Markov type inequality

We say that a compact set $\emptyset \neq E \subset \mathbb{R}^m$ satisfies Markov type inequality (or: is a Markov set) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in \mathcal{P}(\mathbb{R}^m)$ and each $\alpha \in \mathbb{Z}_+^m$,

$$\|D^\alpha P\|_E \leq (C(\deg P)^\kappa)^{|\alpha|} \|P\|_E \,, \tag{1}$$

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$$D^\alpha P = \frac{\partial^{|\alpha|} P}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}} \quad \text{and} \quad |\alpha| = \alpha_1 + \cdots + \alpha_m.$$

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Clearly, by iteration, it is enough to consider in the above definition multi-indices α with $|\alpha| = 1$.

$\mathbb{Z}_+ := \{0, 1, 2, \dots\}$.

A generalization to several variables of a local form of the classical Markov inequality

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Local Markov inequality of exponent σ

We say that a compact set $\emptyset \neq E \subset \mathbb{R}^m$ admits a local Markov inequality of exponent $\sigma \geq 1$ if there are constants $C, \rho > 0$ such that for all polynomials P, $x \in E$ and $0 < \epsilon \leq 1$,

$$|D^\alpha P(x)| \leq (C\epsilon^{-\sigma})^{|\alpha|}(\deg P)^{\rho|\alpha|}\|P\|_{E\cap B(x,\epsilon)} \tag{2}$$

where $B(x, \epsilon)$ denotes the closed ball of radius ϵ centered at x.

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For $E \subset \mathbb{R}^m$, the choice $\epsilon = 1$ in the above form of local Markov inequality immediately yields

$$\|D^\alpha P\|_E \leq (C(\deg P)^\rho)^{|\alpha|}\,\|P\|_E\,,$$

i.e. Markov type inequality (1).

Equivalence of Markov and local Markov inequalities

Theorem (L.P. Bos and P.D Milman, 1995)

Local Markov inequality (2) is equivalent to Markov inequality (1).

The above theorem is a consequence of Theorem E together with Theorem B of the following paper

L.P. Bos, P.D. Milman, *Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains*, Geometric and Functional Analysis 5 (1995), 853–923.

Tangential Markov inequalities

L.P. Bos, N. Levenberg, P. Milman, B.A. Taylor

Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^N$, Indiana Univ. Math. J. 44 (1995) 115–138.

Some notations

Certain subsets of $\mathcal{P}(\mathbb{R}^N)$

Having fixed the dimension N, we define for a natural number $m < N$ and $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$

$$\mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N) = \left\{ P \in \mathcal{P}(\mathbb{R}^N) : P(x) = \sum_{\alpha_1=0}^{d_1} \cdots \sum_{\alpha_m=0}^{d_m} p_{\alpha_1,\ldots,\alpha_m}(\pi_m(x)) x_1^{\alpha_1} \cdots x_m^{\alpha_m} \right\}.$$

Here π_m is the function on $\mathbb{R}^N$ defined by

$$\pi_m((x_1, \ldots, x_N)) = (x_{m+1}, \ldots, x_N).$$

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$\mathcal{P}_{m,\mathbf{d}}-$determining set

We say that $E \subset \mathbb{R}^N$ is a $\mathcal{P}_{m,\mathbf{d}}-$determining set if for each $P \in \mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N)$, $P_{|E} = 0$ implies $D^\alpha P_{|E} = 0$, for all $\alpha \in \mathbb{Z}_+^N$.

Certain algebraic varieties

We will consider algebraic sets $V = V_{m,\mathbf{d}}$ for which there exist a natural number $m < N$ and $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$ such that

$$\forall_{P \in \mathcal{P}(\mathbb{R}^N)} \exists_{\hat{P} \in \mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N)} \quad P_{|V_{m,\mathbf{d}}} = \hat{P}_{|V_{m,\mathbf{d}}}, \tag{3}$$

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Example

$V_{2,(1,1)} = \{(t, s, x, y) \in \mathbb{R}^4 : t^2 = 1 - x^4,\ s^2 = 1 - y^2\}.$

Inspired by the considerations that have been made (see [BK] and [BCCK]) we deal with the following definition

Markov set and Markov inequality on $V_{m,\mathbf{d}}$

Let $V_{m,\mathbf{d}}$ be given as before. Suppose that $V_{m,\mathbf{d}}$ is nonempty. A compact set $\emptyset \neq E \subset V_{m,\mathbf{d}}$ is said to be a $V_{m,\mathbf{d}}$-Markov set if there exist $M, r > 0$ such that

$$\|D^\alpha P\|_E \leq M^{|\alpha|}(\deg P)^{r|\alpha|}\|P\|_E, \quad P \in \mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N), \quad \alpha \in \mathbb{Z}_+^N. \qquad (4)$$

This inequality is called a $V_{m,\mathbf{d}}$-Markov inequality for E.

[BK] M. Baran, A. Kowalska, *Sets with the Bernstein and generalized Markov properties*, Ann. Polon. Math. 111 (3) (2014) 259–270.

[BCCK] L. Białas-Cież, J.P. Calvi, A. Kowalska, *Polynomial inequalities on certain algebraic hypersurfaces*, J. Math. Anal. Appl. 459 (2) (2018) 822–838.

Markov set on $V_{m,\mathbf{d}}$: example

$V_{2,(1,1)} = \{(t,s,x,y) \in \mathbb{R}^4 : t^2 = 1 - x^4,\ s^2 = 1 - y^2\}$

The compact set $E = \{(t,s,x,y) \in V_{2,(1,1)} : (x,y) \in [0,1]^2\}$ is a $V_{2,(1,1)}$-Markov.

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Further examples of $V_{m,\mathbf{d}}$-Markov sets can be given using the following lemma

Lemma

Let $\emptyset \neq E$ be a compact subset of $V_{m,\mathbf{d}}$. If $\pi_m(E)$ is a Markov set (with A and η) and there exist $B, \lambda > 0$ (depending only on E, m and $\mathbf{d}$) such that for every polynomial

$$P = \sum_{\alpha_1=0}^{d_1} \cdots \sum_{\alpha_m=0}^{d_m} p_{\alpha_1,\ldots,\alpha_m}(\pi_m(x)) x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N)$$

$$\|p_{\alpha_1,\ldots,\alpha_m}\|_{\pi_m(E)} \leq B(\deg P)^{\lambda} \|P\|_E, \quad (\alpha_1,\ldots,\alpha_m) \in \mathbb{Z}_+^{m,\mathbf{d}},$$

then E is a $V_{m,\mathbf{d}}$-Markov set.

Local Markov inequality on $V_{m,\mathbf{d}}$

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Let $\emptyset \neq E$ be a compact subset of $V_{m,\mathbf{d}}$. For a fixed $a \in E$ and $\epsilon > 0$ let

$$L_m(a, \epsilon) = \left\{x \in V_{m,\mathbf{d}} : \pi_m(x) \in B(\pi_m(a), \epsilon)\right\},$$

where $B(\pi_m(a), \epsilon)$ denotes the closed ball ($(N-m)$-dimensional) of radius ϵ centered at $\pi_m(a)$.

Local Markov inequality on $V_{m,\mathbf{d}}$

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Local $V_{m,\mathbf{d}}$-Markov inequality

We say that E admits a local $V_{m,\mathbf{d}}$-Markov inequality of exponent $\sigma \geq 1$ if there are constants $C, \rho > 0$ (depending only on E) such that

$$|D^\alpha P(a)| \leq (C\epsilon^{-\sigma})^{|\alpha|}(\deg P)^{\rho|\alpha|} \, \|P\|_{E \cap L_m(a,\epsilon)} \qquad (5)$$

for every $a \in E$, $0 < \epsilon \leq 1$, $P \in \mathcal{P}_{m,\mathbf{d}}(\mathbb{R}^N)$ and $\alpha \in \mathbb{Z}_+^N$.

Equivalence of $V_{m,\mathbf{d}}$-Markov and local $V_{m,\mathbf{d}}$-Markov inequalities

Theorem

Local $V_{m,\mathbf{d}}$-Markov inequality (5) is equivalent to $V_{m,\mathbf{d}}$-Markov inequality (4).

Equivalence of $V_{m,\mathbf{d}}$-Markov and local $V_{m,\mathbf{d}}$-Markov inequalities

Theorem

Local $V_{m,\mathbf{d}}$-Markov inequality (5) is equivalent to $V_{m,\mathbf{d}}$-Markov inequality (4).

The idea of a proof comes from the mentioned paper of Bos and Milman.

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Thank you for your attention!