Geometric Valuation Theory

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Valuations on compact convex sets play a prominent role in geometry. They were critical in Dehn's solution to Hilbert's Third Problem in 1901 (see [4,5]). They are defined as follows. A function Z whose domain is a collection of sets $\mathcal{S}$ and whose co-domain is an Abelian semigroup is called a *valuation* if

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}$.

The first classification result for valuations on the space of compact convex sets, $\mathcal{K}^n$, in $\mathbb{R}^n$ (where $\mathcal{K}^n$ is equipped with the topology induced by the Hausdorff metric) was established by Blaschke.

Theorem 1 (Blaschke)**.** *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and* $\mathrm{SL}(n)$ *invariant valuation if and only if there are constants* $c_0, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$*.*

Here $V_0(K)$ is the Euler characteristic and $V_n(K)$ the n-dimensional volume of $K \in \mathcal{K}^n$.

Probably the most famous result in the geometric theory of valuations is the Hadwiger characterization theorem.

Theorem 2 (Hadwiger)**.** *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation if and only if there are constants* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$*.*

Here $V_0(K), \dots, V_n(K)$ are the intrinsic volumes (Quermassintegrals) of $K \in \mathcal{K}^n$. Hadwiger's theorem shows why the intrinsic volumes are the most basic functionals in Euclidean geometry. Hadwiger's theorem finds powerful applications in Integral Geometry and Geometric Probability (see [4, 5]).

The fundamental results of Blaschke and Hadwiger were the starting point of the development of Geometric Valuation Theory. Classification results for valuations invariant (or covariant) with respect to important groups are central questions. A strengthening of Blaschke's result is obtained by weakening the assumption of continuity to upper semicontinuity.

Theorem 3 ([8])**.** *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is an upper semicontinuous, translation and* $\mathrm{SL}(n)$ *invariant valuation if and only if there exist* $c_0, c_n \in \mathbb{R}$ *and* $c \geq 0$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K) + c\,\Omega(K)$$

for every $K \in \mathcal{K}^n$*.*

The 'new' valuation $\Omega(K)$ in this characterization theorem is Blaschke's *affine surface area* of the convex set K. This functional is fundamental in Affine Differential Geometry. It can be defined by

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}}\,dx,$$

where $\kappa(K,x)$ is the generalized Gaussian curvature of ∂K at x. Since many problems in Discrete and Stochastic Geometry are $\mathrm{SL}(n)$ invariant, affine surface area has found numerous applications in these fields (and even in areas such as image analysis).

The classification of $\mathrm{SL}(n)$ and translation invariant valuations from Theorem 3 is in the tradition of Felix Klein's Erlangen Program: Invariants of transformation groups are classified that additionally are valuations. This point of view also motivated far-reaching extensions of Theorem 3 in joint work with Matthias Reitzner [9] (complemented by Haberl and Parapatits [3]). In addition to affine surface area, also the L_p-affine surface areas introduced by Lutwak [10] are characterized and a new family of functionals, which are now called Orlicz affine surface areas, were introduced. This classification result has become one of the starting points of the Orlicz-Brunn-Minkowski theory (see [11]). The extension of this family of results to rigid motion invariant valuations is a major open problem.

Within Geometric Valuation Theory, convex-set-valued valuations were introduced and classified (the first such result was [6]). These results have applications to isoperimetric inequalities, were extended to lattice polytopes [1], and very recently, to function spaces [2]. Geometric Valuation Theory on function spaces is just emerging (see [7]) and should lead to the identification of the most important geometric functionals on different spaces.

References

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