Summary of recent research
E. Kowalski

In the last few years, my work has been concentrated in three main directions, with some overlap.

Probabilistic models in number theory. (Collaborations with A. Nikeghbali and others) A number of results (such as the Erdös-Kac Theorem) or conjectures (such as the Keating-Snaith conjectures on moments of the Riemann zeta function) in analytic number theory can be interpreted from a probabilistic point of view. Viewed in these terms, they are *non-standard limit theorems*. More precisely, the asymptotic formulas that appear in terms of Fourier analysis can be seen as breaking the universality features inherent in general Central Limit Theorems, and they reveal connections with other probability models.

The formulas that reveal this non-universal behavior suggest fascinating and mysterious links with such objects as random permutations (as above) or random matrices. This leads, in turn, to (partly heuristic) models for basic arithmetic objects. These can be very simple (integers and their prime factorizations) or rather sophisticated (quite general families of automorphic forms and representations on reductive groups over global fields).

In developing the insights arising from these observations, substantial links with probability theory appear. Indeed, collaborations with A. Barbour and F. Delbaen, and further independent work, has shown that the type of behavior seen in these arithmetic results also appears in many purely probabilistic contexts. This leads, for instance, to a powerful method for Poisson approximation, as well as very general local limit theorems.

Arithmetic applications of expander graphs. (Collaborations with J. Ellenberg and C. Hall, F. Jouve and D. Zywina) Expander graphs are families of finite graphs that are simultaneously relatively sparse (bounded degree typically) and highly-connected (measured by the Cheeger constant, or the size of the first non-zero Laplace eigenvalue). The study of these graphs has been deeply connected with number theory and harmonic analysis almost since their original discovery, and recent years have led to new significant and surprising applications of these graphs in number theory, as well as in other areas such as geometry and group theory. My work concerns two types of applications:

(1) The study of the "generic" behavior of elements of a finitely-generated subgroup Γ of some arithmetic group $\mathbf{G}$ depends essentially on expansion properties of the Cayley graphs of congruence subgroups of Γ. With F. Jouve and D. Zywina, I have obtained many results in this direction, a prototypical example being the determination of the "generic" Galois of elements of arithmetic groups. In the split connected case, this is the Weyl group of the underlying algebraic group. (Such questions have been further developed by Gorodnik–Nevo, Lubotzky–Rosenzweig and Lubotzky–Meiri).

(2) Very strong arithmetic applications are obtained concerning finiteness of rational points of bounded degree on algebraic curves over number fields, using the same type of expansion properties of finite linear groups. The crucial ingredients are the Faltings–Frey Theorem and a differential-geometric inequality of Li and Yau, which leads to a strong lower-bound for the gonality of a sequence of étale coverings $U_n \longrightarrow U$ of a fixed algebraic curve $U/\mathbf{C}$, provided only that the associated Cayley-Schreier graphs (relative to a fixed finite generating set of $\pi_1(U)$) form an expanding family (or even satisfies a slightly weaker expansion property).

These applications depend on recent results of Helfgott, Breuillard–Green–Tao and especially of Pyber–Szabó concerning expansion of finite linear algebraic groups, even when these are not obtained by reduction of an arithmetic group or of a Zariski-dense subgroup of an arithmetic group.

Analytic properties and applications of trace functions. (Collaborations with É. Fouvry, Ph. Michel and W. Sawin) Much of my research during the last years has been dedicated to developing with É. Fouvry, Ph. Michel and W. Sawin (as well as our students) the analytic theory and applications of *trace functions* of ℓ-adic sheaves, building on the algebraic and geometric properties of these objects, as defined and studied especially by Deligne, Grothendieck, Katz and Laumon. The crucial points for the applications to analytic number theory are

(1) the very general form of Deligne's Riemann Hypothesis can be interpreted as a strong form of "quasi-orthogonality" of geometrically irreducible trace functions, which is a very powerful analytic tool;
(2) the dependency with respect to the underlying prime p, which is essential in analytic applications, can be kept under control analytically using a single complexity parameter, the "analytic conductor" of the underlying sheaves.

The result we have obtained up to now include:

- A proof that trace functions do not correlate against Hecke eigenvalues of classical modular forms, with consequences concerning the equidistribution of "twisted" discrete horocycles; here the sheaf-theoretic Fourier transform of Deligne, Katz and Laumon plays an essential role. Moreover, following ideas of Holowinsky–Nelson, this has been extended to modular forms on GL_3 with Y. Lin.
- Building on this, a proof that trace functions (with some necessary exceptions) do not correlate against the primes, or against the Möbius function. These results, in turn, have many applications and go well-beyond previously known results.
- A proof that trace functions satisfy a very strong and explicit form of "inverse theorem" for Gowers norms; this provides, in particular, the first examples of explicit functions such that their Gowers norms of all order are as small as those of "random" functions.
- An estimate, using spherical codes, of the number of trace functions with bounded complexity.
- Estimates for bilinear forms of a general type

  $$\sum_{m,n} \alpha_m \beta_n \mathrm{Kl}_k(amn; p)$$

  with hyper-Kloosterman sums as coefficients, giving a non-trivial bound in the crucial case where m and n can be smaller than the range where straightforward Fourier-theoretic methods are applicable. This leads to the solution of the long-standing open problem of the asymptotic formula for

  $$\sum_{\chi} \overline{L(g \times \chi, 1/2)} L(f \times \chi, 1/2)$$

  when f and g are both cusp forms.