SIMPLE AMENABLE OPERATOR ALGEBRAS

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Operator algebras arise concretely as certain subalgebras of the collection, $\mathcal{B}(\mathcal{H})$, of bounded or continuous operators on a Hilbert space $\mathcal{H}$. In the case when $\mathcal{H}$ is finite dimensional, $\mathcal{B}(\mathcal{H})$ is the algebra of $n \times n$ complex matrices. We have both algebraic and analytic structures on $\mathcal{B}(\mathcal{H})$. In addition to the vector space operations, multiplication of operators is defined by composition, so is in general non-commutative. There is also a compatible involution: the adjoint operation *, which from the matrix view point, is given by conjugate transpose. This algebraic structure is combined with analytic information coming from the norm topology and weak operator topology of pointwise convergence.

This gives rise to two fundamental classes of operator algebras: C^*-algebras and von Neumann algebras, depending on whether one asks for *subalgebras of $\mathcal{B}(\mathcal{H})$ which are closed in the norm or weak operator topologies on $\mathcal{B}(\mathcal{H})$ respectively. At first sight this distinction looks quite minor, but leads to quite different behaviours. This is seen in the abelian case: commutative C^*-algebras consist of continuous functions vanishing at infinity on a locally compact space, while commutative von Neumann algebras are formed from essentially bounded functions on a measure space. These flavours of topology and measure theory persist into the non-commutative setting. Concepts such as dimension and invariants such as K-theory pervade the study of C^*-algebras, while measure-theoretic style methods are often used to study von Neumann algebras. Examples arise naturally from mathematical objects such as groups and dynamics, with key notions such as amenabilty being captured at the operator algebraic level.

A major revolution in operator algebras took place in the 1970's driven by Alain Connes' celebrated work on the structure and classification of amenable von Neumann factors (factors are the fundamental building blocks of von Neumann algebras). In particular, he showed that amenable von Neumann algebras have striking internal approximations by finite dimensional algebras ([3]). This led to the complete classification of amenable von Neumann von Neumann algebras (and an analogous result in the setting of measurable dynamics), with the final piece of the puzzle being completed by Haagerup ([7]), and has been instrumental ever since. Connes theorem underpins both Jones' dramatic work on subfactors, and the repeated major breakthroughs made in the structure of non-amenable von Neumann algebras over the last 20 years powered by Popa's deformation-rigidity theory.

In the topological setting, George Elliott classified those C^*-algebras with the analogous internal approximation property in the 1970's in terms of operator algebraic K-theory ([4]). However, unlike the von Neumann setting, there are topological obstructions to internal finite dimensional approximations and many naturally occurring amenable C^*-algebras, for example those associated to irrational rotations on the circle, fail to enjoy this property. Indeed, even in the commutative

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situation having good internal finite dimensional approximations is relatively rare; for continuous functions on a compact Hausdorff space X such approximations can only be found when X is zero dimensional. Moreover, abstractly characterising the existence of internal finite dimensional approximations seems very challenging, as no potential candidate has been identified. However, complementary dramatic developments in the 90's by Elliott and Kirchberg provided classification results for certain subclasses of amenable C^*-algebras without finite dimensional internal approximations. These results motivated what became a major research activity in C^*-algebras over the next 25 years — the Elliott classification programme — to determine exactly which amenable C^*-algebras can be reasonably classified by K-theoretic data?

Exotic examples of Villadsen, Rørdam and Toms constructed in the 2000's show that a complete classification of all simple amenable C^*-algebras by reasonably computable invariants of a K-theoretic nature is impossible (see [11]). Topological spaces of very rapidly growing dimensions play a crucial role in these constructions, and the resulting C^*-algebras should be viewed as having infinite topological covering dimension. This can now be made precise using Winter and Zacharias's *nuclear dimension*. This is a non-commutative extension of Lebesgue covering dimension from locally compact Hausdorff spaces to amenable C^*-algebras. Moreover, combining the work of vast numbers of researchers worldwide, whether the nuclear dimension is finite or infinite now provides the dividing line between the classifiable and exotic: Simple separable C^*-algebras are classified by K-theory and traces almost[1] precisely when they have finite non-commutative covering dimension in this sense ([10, 5, 6]). With hindsight the need for an extra dimensional condition in the C^*-setting which is not present in Connes theorem makes sense: measure theory doesn't see dimension.

In the light of the classification theorem, a major goal is to find methods for determining the nuclear dimension of simple nuclear C^*-algebras. While this can now be done concretely in a number of relatively straightforward examples, in more complicated cases, particularly those coming from actions of discrete amenable groups on compact Hausdorff spaces, this is a challenging task. However, Toms and Winter noticed that in a particular class of inductive limit examples (which includes some of the exotic counterexamples to classification) finite nuclear dimension occurs simultaneously with other properties of very different natures,[2] leading to the conjecture that finite nuclear dimension ought to be abstractly characterisable through these alternative view points. At the time it was made, the conjecture was quite bold, but developments have shown it to be very well founded ([8, 9, 1, 2]). I now view finite nuclear dimension as one facet of an underlying meta-notion of regularity for simple amenable C^*-algebras, and in different examples we will obtain this regularity in different ways.

Interestingly, with hindsight one can now see remarkable parallels between the Toms-Winter conjecture, and aspects of Connes theorem. This happens at a high level: the concepts appearing in the Toms-Winter conjecture resemble major landmarks in Connes work, and also at the level of arguments: the adoption of proof

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[1]'Almost' refers to an additional Universal Coefficient Theorem condition, which holds in all known examples (though verifying it in general is a major challenge). Describing this property here through, would take us too far afield.

[2]$\mathcal{Z}$-stability and strict comparison, though I won't describe these properties here.

techniques from one setting to another, and the direct use of Connes results applied to von Neumann algebras constructed from C^*-algebras. Given the influence Connes theorem has had on future developments in von Neumann algebras and beyond, this sets the stage for an exciting next phase in the theory of C^*-algebras as we explore these parallels in other contexts.

References

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