Continuation of a quaternionic logarithm (G. Gentili J. Prezelj and F. Vlacci)

Formulation of the problem

The mapping $\exp : \mathbb{H} \to \mathbb{H} \setminus \{0\}$ is surjective but not a covering map. If $\gamma : [0,1] \to \mathbb{H} \setminus \{0\}$ is a continuous curve, is there a continuous curve $\log\gamma := \ln|\gamma| + \mathrm{Arg}(\gamma) \to \mathbb{H}$, where $\mathrm{Arg}(\gamma) : [0,1] \to \mathrm{Im}(\mathbb{H})$ and $\exp(\log\gamma) = \gamma$? If γ is a closed curve, can the winding number be defined?

Basic definitions

- $\mathbb{S} = \{q \in \mathbb{H}, q^2 = -1\}$;
- $\exp(k\pi\mathbb{S}) = (-1)^k$;
- $\mathbb{RP}^2 \simeq \mathbb{S}/_{q\sim -q}$;

- $\gamma : [0,1] \to \mathbb{H} \setminus \{0\}$: continuous curve which intersects the real axis finitely many times, $\gamma(0) \notin \mathbb{R}$; hence there exists unique real numbers $x(0), y(0)$, $y(0) > 0$ and an imaginary unit $I(0)$ such that $\gamma(0) = x(0) + I(0)y(0)$;

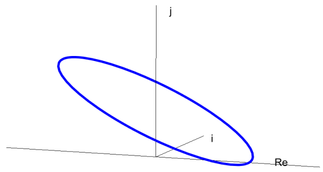
- γ is *very tame* if $\exists$ a continuous projective curve $\mathcal{J}_\gamma : [0,1] \to \mathbb{RP}^2$; such that $\gamma(t) \in \mathbb{R} + J(t)\mathbb{R}, [J(t)] = \mathcal{J}_\gamma(t)$ for every $t \in [0,1]$.
- γ is a *very tame closed* curve if it is very tame, closed and the projective curve $\mathcal{J}_\gamma$ is also closed;
- if γ is very tame, then $\exists \mathcal{I}_\gamma : [0,1] \to \mathbb{S}$ continuous, $[\mathcal{I}_\gamma] = \mathcal{J}_\gamma$, $\mathcal{I}_\gamma(0) = I(0)$;
- a very tame closed curve γ is *twisted* if $\mathcal{I}(0) = -\mathcal{I}(1)$.

- define $\arg(\gamma) : [0,1] \to \mathbb{R}$ by $\mathcal{I}_\gamma \arg(\gamma) := \mathrm{Arg}(\gamma)$;
- define the complex curve $\varphi_\gamma := |\gamma| \exp(i\arg(\gamma))$;
- if γ is closed then the *winding number* is defined as $\omega(\gamma, 0) := \omega(\varphi_\gamma, 0)$;

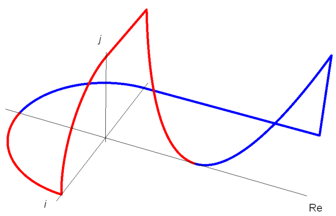
Examples

- $\gamma(t) = \cos(2\pi t) + i\sin(2\pi t)$, $t \in [0,1]$, is a closed very tame not twisted curve; $\omega = 1$;

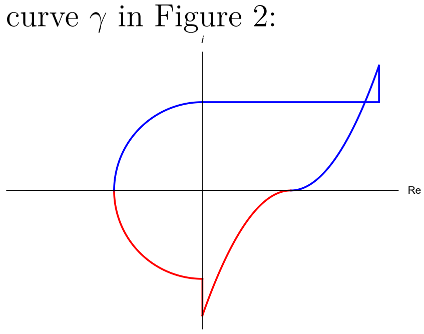
- Figure 1: a very tame twisted curve, $\omega = 0$:



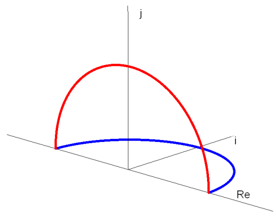
- Figure 2: a very tame twisted curve; $\omega = 1$:



- Figure 3: the curve φ_γ for the curve γ in Figure 2:

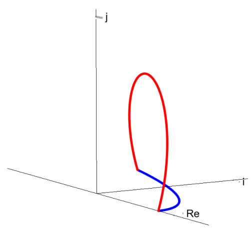


- any curve $\gamma : [0,1] \to \mathbb{H} \setminus \mathbb{R}$ is a very tame curve and if closed is a very tame closed curve;
- a curve which is not very tame:



- if $\chi : \mathbb{R} \to [0,1]$ is a $\mathcal{C}^\infty$-increasing function satisfying $\chi((-\infty, 0]) = 0$ and $\chi([1,\infty)) = 1$, then the curve
$$\gamma(t) = t+1+(t-1/2)(i\chi(t-1/2) + j\chi(1/2-t)), t \in [0,1]$$
is not very tame at $t = 1/2$ but the choice of the argument 0 at the point $\gamma(1/2) = 3/2$ guarantees the existence of a continuous function $\log(\gamma)$;

- a curve which is not tame if $x(1/2) < 0$ and tame if $x(1/2) > 0$: $\gamma(t) = x(t) + (t-1/2)\cdot(i\cos(1/(t-1/2)) + j\sin(1/(t-1/2))), t \in [0,1]$;
- a curve which is very tame, a tame closed curve but not a very tame closed curve:



Main result 1

If γ is a very tame curve then $\log\gamma$ always exists.

Main Result 2

$$\gamma \text{ very tame closed not twisted: } 2\pi I(0)\omega(\gamma,0) = \log\gamma(1) - \log\gamma(0)$$
$$\gamma \text{ very tame closed twisted: } 2\pi I(0)\omega(\gamma,0) = -\log\gamma(1) - \log\gamma(0).$$